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On the First Cartan Invariants in Characteristic 2 of the Groups $SL_3(2^m)$ and $SU_3(2^m)$

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1. INTRODUCTION

In [2, Theorem B] some information was obtained about the first Cartan invariants in characteristic 2 of the groups $SL_3(2^m)$ and $SU_3(2^m)$. The purpose of this paper is to obtain an explicit formula for this invariant.

THEOREM. *Let $G = SL_3(2^m)$ or $SU_3(2^m)$, and $\Phi_{(\phi,\phi)}$ the projective indecomposable character of G associated with the trivial irreducible Brauer character for the prime $p = 2$. Then*

$$\begin{aligned} C^{(m)} &= \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)} \rangle \\ &= 7^m + \varepsilon^m - 2^{3m+1} + a^m + b^m \\ &\quad + [2^m + c^m + d^m] + [(\varepsilon 2)^m + (\varepsilon c)^m + (\varepsilon d)^m] \\ &\quad + [e^m + f^m] + [(\varepsilon e)^m + (\varepsilon f)^m] \\ &\quad - 2[\alpha^m + \beta^m + \gamma^m] - 2[(\varepsilon\alpha)^m + (\varepsilon\beta)^m + (\varepsilon\gamma)^m], \end{aligned}$$

where $\varepsilon = 1$ if $G = SU_3(2^m)$, $\varepsilon = -1$ if $G = SL_3(2^m)$, a, b are two roots of $x^2 - 12x + 24 = 0$, c, d are two roots of $x^2 - 8x - 8 = 0$, e, f are two roots of $x^2 - 8x + 10 = 0$, and α, β, γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$.

COROLLARY.

$$\lim_{m \to \infty} \frac{C^{(m)}}{(6 + 2\sqrt{3})^m} = 1.$$

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We will prove the theorem in Sections 6–8 for $G = SL_3(2^m)$ and in Section 10 for $G = SU_3(2^m)$.

In [2], the formulas for the degrees of the projective indecomposable characters in characteristic 2 of the groups $SL_3(2^m)$ and $SU_3(2^m)$ were obtained. In Sections 5 and 9 we will give another method to obtain these formulas. Analogous argument can be used to study projective characters in characteristic 2 (resp. 3) of the groups $G_2(2^m)$ (resp. $Ree(3^m)$ and $G_3(3^n)$). See [5, 6].

Results of this sort were obtained for the groups $Suz(2^m)$ and $Sp_4(2^n)$ in [1].

This paper is inspired by [1, 2]. The above formula is simplified by Professor Walter Feit. Without his modification the formula would be more complicated. This paper is part of the author's thesis [4] under the supervision of Professor Walter Feit. For other parts see [5, 6].

2. Notation

The following notation will be used throughout this paper: m is a fixed natural number, n is an arbitrary natural number unless otherwise stated, $G(n) = SL_3(2^n)$, $q = 2^m$, K is the algebraic closure of the finite field of order 2, $G_\infty = SL_3(K)$, Z: the set of rational integers, Z^+: the set of nonnegative integers, N: the set of positive integers, $S(n) = Z/nZ$, A is an arbitrary subset of $S(n)$ unless otherwise stated, and ϕ is the empty set.

For $I \subset A$, we identify I with its characteristic function in A, i.e., for all $k \in A$,

$$\begin{aligned} I(k) &= 1, &&\text{if } k \in I,\\ &= 0, &&\text{if } k \in A \setminus I; \end{aligned}$$

$\Omega(n, A) = \{I = (I_1, I_2) | I_1, I_2 \subset A\}$. For $k \in A$ and $I = (I_1, I_2) \in \Omega(n, A)$, we define $I(k) = (I_1(k), I_2(k))$.

In $\Omega(n, A)$, we define two operations: for $I = (I_1, I_2)$, we define $I' \in \Omega(n, A)$ as

$$\begin{aligned} I'(k) &= (1, 1), &&\text{if } I(k) = (0, 0),\\ &= (1, 0), &&\text{if } I(k) = (1, 0),\\ &= (0, 1), &&\text{if } I(k) = (0, 1),\\ &= (0, 0), &&\text{if } I(k) = (1, 1), \end{aligned}$$

for all $k \in A$. For $I = (I_1, I_2)$, we define $\bar{I} = (I_2, I_1)$.

In case $A = S(n)$, we write $\Omega(n)$ instead of $\Omega(n, S)$. In case $n = m$, we omit the n in all of the above notation. That is, $G = G(m)$, $S = S(m)$, $\Omega(A) = \Omega(m, A)$, and $\Omega = \Omega(m)$. $T = (S, S) \in \Omega$. Hence $T' = (\phi, \phi)$. For $I = (I_1, I_2) \in \Omega$, x, $y \in \{0, 1\} \subset Z$, let $N(I, (x, y)) = |\{k \in S \mid I(k) = (x, y)\}|$. 1_G: the identity element of G, σ: the Frobenius automorphism of G or G_∞. If g, h are complex-valued class functions defined on G, let

$$\langle g, h \rangle = \frac{1}{|G|} \sum_{x \in G} g(x) \overline{h(x)}.$$

Γ is the Steinberg character of G. If g is a class function on G, let $g^\sigma = g \circ \sigma$.

Let H be a subgroup of G_∞. If M is a KH-module, then M^σ is the module obtained by letting $h \in H$ act on M^σ as h^σ acts on M. Let M^* denote the contragredient module. If H is finite and ρ is the Brauer character afforded by M, then M^* affords $\bar{\rho}$, where the bar denotes complex conjugation.

Lie algebra representations are over the complex number field. Modular representations of G_∞ or G are over K. We do not distinguish the modular representation and its Brauer character.

3. The Irreducible Brauer Characters

Let $\mathfrak{g}$ be the simple Lie algebra of type A_2 and w_1, w_2 be fundamental weights of $\mathfrak{g}$ associated with simple roots α_1, α_2 as in Fig. 1. Let $V(0)$, $V(w_1)$, $V(w_2)$, and $V(w_1 + w_2)$ be the irreducible $\mathfrak{g}$-modules with highest weights 0, w_1, w_2, and $w_1 + w_2$, respectively. By [2], these give the corresponding irreducible G_∞-modules in characteristic 2, and their dimensions are

$$\dim V(0) = 1, \dim V(w_1) = \dim V(w_2) = 3, \dim V(w_1 + w_2) = 8.$$

Let $\rho(0) = 1$, $\rho(w_1)$, $\rho(w_2)$, and $\rho(w_1 + w_2)$ denote the Brauer characters of G

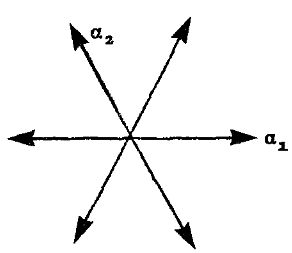

Fig. 1. Root system of type A_2.

afforded by $V(0)$, $V(w_1)$, $V(w_2)$, and $V(w_1+w_2)$, respectively. For $I=(I_1, I_2) \in \Omega$, define

$$\rho_I = \prod_{i \in S} \rho(I_1(i)\, w_1 + I_2(i)\, w_2)^{\sigma^i}. \tag{3.1}$$

By the tensor product theorem [8], $\mathfrak{B} = \{\rho_L \mid L \in \Omega\}$ is the set of all irreducible Brauer characters of G. Furthermore, $\rho_T = \Gamma$ is the Steinberg character of G and $\rho_{(\phi,\phi)} = 1$ is the trivial character.

By [2] or computing the weight spaces of $F(w_1)$, $F(w_2)$, and $F(w_1+w_2)$, we get Lemmas 3.2 and 3.3.

LEMMA 3.2.

$$\begin{aligned}
\rho(w_1)^2 &= \rho(w_1)^\sigma + 2\rho(w_2),\\
\rho(w_2)^2 &= \rho(w_2)^\sigma + 2\rho(w_1),\\
\rho(w_1)\,\rho(w_2) &= \rho(w_1+w_2) + 1,\\
\rho(w_1)\,\rho(w_1+w_2) &= \rho(w_2)\,\rho(w_1)^\sigma + 2\rho(w_2)^\sigma + 3\rho(w_1),\\
\rho(w_2)\,\rho(w_1+w_2) &= \rho(w_1)\,\rho(w_2)^\sigma + 2\rho(w_1)^\sigma + 3\rho(w_2),\\
\rho(w_1+w_2)^2 &= 2\rho(w_1+w_2) + \rho(w_1+w_2)^\sigma + 2\rho(w_1)\,\rho(w_1)^\sigma\\
&\quad + 2\rho(w_2)\,\rho(w_2)^\sigma + 4.
\end{aligned}$$

We note that the last three equations can be derived from the first three equations.

LEMMA 3.3. $\overline{\rho(w_1)} = \rho(w_2)$, $\overline{\rho(w_1+w_2)} = \rho(w_1+w_2)$. *Furthermore,* $\bar{\rho}_I = \rho_{\bar{I}}$ *for all* $I \in \Omega$.

THEOREM 3.4. *The Grothendieck ring $R(G)$ of G in characteristic 2 is isomorphic to the commutative Z-algebra which is generated by elements x_i, y_i, z_i, $i \in S$ that satisfy*

$$x_i^2 = x_{i+1} + 2y_i, \qquad y_i^2 = y_{i+1} + 2x_i, \qquad x_i y_i = z_i + 1.$$

Proof ([1, Theorem 3.5; 5, 6, Theorem 3.4]). Let D denote the Z-algebra defined in the statement. By Lemma 3.2 there exists an epimorphism $p: D \to R(G)$. By the tensor product theorem [8], $R(G)$ has rank 4^m as a Z-module. D is generated as a Z-module by the elements $\prod_{i \in S} r_i$, where $r_i = 1, x_i, y_i$, or z_i. Hence D has Z-rank at most 4^m. This implies that p is an isomorphism.

4. The Projective Indecomposable Characters

Define

$$\eta(0)=1, \qquad \eta(w_1)=\rho(w_1), \qquad \eta(w_2)=\rho(w_2),$$
$$\eta(w_1+w_2)=\rho(w_1+w_2)-2. \tag{4.1}$$

For $I=(I_1,I_2)\in\Omega$, define

$$\eta_I=\prod_{i\in S}\eta(I_1(i)\,w_1+I_2(i)\,w_2)^{\sigma^i}. \tag{4.2}$$

We note that η_I are similar to ψ's defined in [3]. By Lemma 3.3, $\overline{\eta_I}=\eta_{\bar{I}}$. Let Φ_I be the principal indecomposable character of G associated with ρ_I. As $\Gamma\rho_I$ is an integral combination of the Φ_J's, so is $\Gamma\eta_I$. We also know that the multiplicity of Φ_J in $\Gamma\eta_I$ is $\langle\Gamma\eta_I,\rho_I\rangle=\langle\Gamma,\eta_{\bar{I}}\rho_J\rangle$ for all I and J in Ω. The fact that

$$\langle\Gamma,\rho_J\rangle=\delta_{T,J} \tag{4.3}$$

is of basic importance for whole computation in this paper.

Definition 4.4. In the Grothendieck ring $R(G)$, we define a length function $l: R(G)\to Z^+$ as follows: for the basis element ρ_I, $I=(I_1,I_2)\in\Omega$, we define

$$l(\rho_I)=\sum_{k\in S}(I_1(k)+I_2(k))=|I_1|+|I_2|.$$

In general, for

$$\mu=\sum_{I\in\Omega}a(\mu,I)\,\rho_I\in R(G)$$

with $a(\mu,I)\in Z$, we define $l(\mu)=\max\{l(\rho_I)|I\in\Omega,\ a(\mu,I)\neq 0\}$.

Lemma 4.5. *Let*

$$\mu=\sum_{I\in\Omega}a(\mu,I)\,\rho_I\in R(G)$$

with $a(\mu,I)\in Z$. Then

(i) $0\leqslant l(\mu)\leqslant 2m$, $l(\mu)=2m$ *if and only if* $a(\mu,T)\neq 0$.

(ii) $\langle\Gamma,\mu\rangle=a(\mu,T)$. *Furthermore,* $\langle\Gamma,\mu\rangle\neq 0$ *if and only if* $l(\mu)=2m$.

Proof. The proof of (i) follows from Definition 4.4 and (ii) from (4.3).

LEMMA 4.6. (i) $l(1)=0$, $l(\rho(w_1))=l(\rho(w_2))=1$, $l(\rho(w_1+w_2))=2$.

(ii) $l(\rho(w_1)^2)=l(\rho(w_2)^2)=1<2l(\rho(w_1))=2l(\rho(w_2))$, $l(\rho(w_1)\,\rho(w_2))=2=l(\rho(w_1))+l(\rho(w_2))$, $l(\rho(w_1)\,\rho(w_1+w_2))=l(\rho(w_2)\,\rho(w_1+w_2))=2<l(\rho(w_1))+l(\rho(w_1+w_2))=l(\rho(w_2))+l(\rho(w_1+w_2))$, $l(\rho(w_1+w_2)^2)=2<2l(\rho(w_1+w_2))$.

Proof. By Lemma 3.2 and Definition 4.4, we get these results.

LEMMA 4.7. *Suppose* $\mu, \xi \in R(G)$. *Then*

(i) $l(\mu^\sigma)=l(\mu)$.

(ii) $l(\mu\xi)\leqslant l(\mu)+l(\xi)$ (*the triangle inequality*).

Proof. The proof of (i) is trivial; (ii) follows from Lemma 4.6 and induction on $l(\xi)+l(\mu)$.

LEMMA 4.8. *Let* $x_1, y_1, x_2, y_2 \in \{0, 1\} \subset Z$. *Then*

$$l(\overline{\eta(x_1w_1+y_1w_2)}\,\rho(x_2w_1+y_2w_2))\leqslant 2.$$

The equality holds exactly in the following cases:

$$\begin{aligned}
\overline{\eta(0)}\,\rho(w_1+w_2)&=\rho(w_1+w_2),\\
\overline{\eta(w_1)}\,\rho(w_1)&=\rho(w_1+w_2)+\cdots,\\
\overline{\eta(w_1)}\,\rho(w_1+w_2)&=\rho(w_1)\,\rho(w_2)^\sigma+\cdots,\\
\overline{\eta(w_2)}\,\rho(w_2)&=\rho(w_1+w_2)+\cdots,\\
\overline{\eta(w_2)}\,\rho(w_1+w_2)&=\rho(w_2)\,\rho(w_1)^\sigma+\cdots,\\
\overline{\eta(w_1+w_2)}\,\rho(0)&=\rho(w_1+w_2)+\cdots,\\
\overline{\eta(w_1+w_2)}\,\rho(w_1)&=\rho(w_2)\,\rho(w_1)^\sigma+\cdots,\\
\overline{\eta(w_1+w_2)}\,\rho(w_2)&=\rho(w_1)\,\rho(w_2)^\sigma+\cdots,\\
\overline{\eta(w_1+w_2)}\,\rho(w_1+w_2)&=2\rho(w_1)\,\rho(w_1)^\sigma+2\rho(w_2)\,\rho(w_2)^\sigma\\
&\quad+\rho(w_1+w_2)^\sigma+\cdots.
\end{aligned}$$

In the above expression, the omitted terms have length 0 *or* 1.

Proof. The proof is nothing but the computation using Lemma 3.2 and (4.1).

In the rest of this section, we fix $I=(I_1, I_2)$, $J=(J_1, J_2)\in\Omega$. unless otherwise stated. By (3.1) and (4.2),

$$\eta_{\bar{I}}\rho_J=\prod_{k\in S}[\eta(I_2(k)\,w_1+I_1(k)\,w_2)\,\rho(J_1(k)\,w_1+J_2(k)\,w_2)]^{\sigma^k}. \tag{4.9}$$

Let $\mu\colon S\to R(G)$ be a function such that, for all $k\in S$, $\mu(k)$ is a term which appears in the expansion for

$$\eta(I_2(k)\,w_1+I_1(k)\,w_2)\,\rho(J_1(k)\,w_1+J_2(k)\,w_2)$$

expressed in terms of the basis elements in $\mathfrak{B}=\{\rho_L\,|\,L\in\Omega\}$. Thus, $\mu(k)$ is a term of the form $x\rho_L$, with suitable coefficient $x\in Z$. By Lemma 4.8, $l(\mu(k))\leqslant 2$. Let

$$[\mu]=\prod_{k\in S}\mu(k)^{\sigma^k}.$$

It is trivial to see that

$$\langle\Gamma, \eta_{\bar{I}}\rho_J\rangle=\sum\langle\Gamma, [\mu]\rangle,$$

where the summation ranges over all possible μ defined above. Of course, we can also let the summation range over all such μ with $\langle\Gamma, [\mu]\rangle\neq 0$, i.e., $l([\mu])=2m$ by Lemmas 4.5 and 4.8. At this moment, we suppose $l([\mu])=2m$ and see what happens. By the triangle inequality and Lemma 4.7, we get

$$l\left(\prod_{k\in A}\mu(k)^{\sigma^k}\right)=2\,|A| \tag{4.10}$$

for all subsets A of S. In particular, for all $k\in S$, we have $l(\mu(k))=2$. By Lemma 4.8,

$$\mu(k)=\rho(w_1+w_2),\qquad \rho(w_1)\,\rho(w_2)^\sigma,\qquad \rho(w_2)\,\rho(w_1)^\sigma,\qquad 2\rho(w_1)\,\rho(w_1)^\sigma,$$
$$2\rho(w_2)\,\rho(w_2)^\sigma,\qquad \text{or}\qquad \rho(w_1+w_2)^\sigma.$$

Moreover, Lemma 4.8 implies

LEMMA 4.11. *Let $k\in S$. Then*

(i) $\mu(k)=\rho(w_1+w_2)$ *if and only if* $I(k)=J'(k)$.

(ii) $\mu(k)=\rho(w_1)\,\rho(w_2)^\sigma$ *if and only if*
(α) $I(k)=(1,0)$, $J(k)=(1,1)$ *or* (β) $I(k)=(1,1)$, $J(k)=(0,1)$.

(iii) *$\mu(k) = \rho(w_2)\rho(w_1)^\sigma$ if and only if* (γ) $I(k) = (0, 1)$, $J(k) = (1, 1)$ *or* (β) $I(k) = (1, 1)$, $J(k) = (1, 0)$.

(iv) *$\mu(k) = 2\rho(w_1)\rho(w_1)^\sigma$, $2\rho(w_2)\rho(w_2)^\sigma$, or $\rho(w_1 + w_2)^\sigma$ if and only if* $I(k) = J(k) = (1, 1)$.

The Lemma 4.11 gives

LEMMA 4.12. *If $\langle \Gamma\eta_I, \rho_J \rangle \neq 0$, then $I(k) + J(k) \geqslant 2$ for all $k \in S$.*

LEMMA 4.13. *If $\mu(j) = \rho(w_1 + w_2)$ for some $j \in S$, then $\mu(k) = \rho(w_1 + w_2)$ for all $k \in S$. Inparticular, $I = J'$.*

Proof. It suffices to show that $\mu(j-1) = \rho(w_1 + w_2)$. Suppose $\mu(j-1) \neq \rho(w_1 + w_2)$. Then $\mu(j-1) = \rho(w_1)\rho(w_2)^\sigma$, $\rho(w_2)\rho(w_1)^\sigma$, $2\rho(w_1)\rho(w_1)^\sigma$, $2\rho(w_2)\rho(w_2)^\sigma$, or $\rho(w_1 + w_2)^\sigma$. In each case, we can easily show that

$$l(\mu(j-1)^{\sigma^{j-1}} \mu(j)^{\sigma^j}) = l(\mu(j-1)\mu(j)^\sigma) < 4,$$

a contradiction to (4.10). Therefore $\mu(j-1) = \rho(w_1 + w_2)$. The last statement follows from Lemma 4.11(i).

LEMMA 4.14. *If $\mu(j) = \rho(w_1 + w_2)^\sigma$ for some $j \in S$, then $\mu(k) = \rho(w_1 + w_2)^\sigma$ for all $k \in S$. In particular, $I = J = T$.*

Proof. It suffices to show that $\mu(j+1) = \rho(w_1 + w_2)^\sigma$. Suppose $\mu(j+1) \neq \rho(w_1 + w_2)^\sigma$. Then $\mu(j+1) = \rho(w_1)\rho(w_2)^\sigma$, $\rho(w_2)\rho(w_1)^\sigma$, $2\rho(w_1)\rho(w_1)^\sigma$, $2\rho(w_2)\rho(w_2)^\sigma$, or $\rho(w_1 + w_2)$. In each case, we can easily show that

$$l(\mu(j)^{\sigma^j} \mu(j+1)^{\sigma^{j+1}}) = l(\mu(j)\mu(j+1)^\sigma) < 4,$$

a contradiction to (4.10). Therefore $\mu(j+1) = \rho(w_1 + w_2)^\sigma$. The last statement follows from Lemma 4.11(iv).

LEMMA 4.15. *Let $j \in S$.*

(i) *If $\mu(j) = \rho(w_1)\rho(w_2)^\sigma$, then $\mu(j+1) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$.*

(ii) *If $\mu(j) = \rho(w_2)\rho(w_1)^\sigma$, then $\mu(j+1) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$.*

(iii) *If $\mu(j) = 2\rho(w_1)\rho(w_1)^\sigma$, then $\mu(j+1) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$.*

(iv) *If $\mu(j) = 2\rho(w_2)\rho(w_2)^\sigma$, then $\mu(j+1) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$.*

PROPOSITION 4.16. *Let* $I=(I_1,I_2)$, $J=(J_1,J_2)\in\Omega$. *For each* $k\in S$, *fix a term* $\mu(k)$ *which appears in the expansion for*

$$\eta(I_2(k)\,w_1+I_1(k)\,w_2)\,\rho(J_1(k)\,w_1+J_2(k)\,w_2)$$

expressed in terms of the basis elements in $\mathfrak{B}=\{\rho_L\mid L\in\Omega\}$. *Let*

$$[\mu]=\prod_{k\in S}\mu(k)^{\sigma^k}.$$

Then $\langle\Gamma,[\mu]\rangle\neq 0$, *i.e.*, $l([\mu]=2m$ *if and only if* $\mu(k)$ *belongs to* $\{\rho(w_1+w_2)$, $\rho(w_1)\,\rho(w_2)^\sigma$, $\rho(w_2)\,\rho(w_1)^\sigma$, $2\rho(w_1)\,\rho(w_1)^\sigma$, $2\rho(w_2)\,\rho(w_2)^\sigma$, $\rho(w_1+w_2)^\sigma\}$ *for all* $k\in S$ *and one of the following holds*:

(i) $I=J'$ *and* $\mu(k)=\rho(w_1+w_2)$ *for all* $k\in S$.

(ii) $I=J=T$ *and* $\mu(k)=\rho(w_1+w_2)^\sigma$ *for all* $k\in S$.

(iii) *Neither* (i) *or* (ii) *holds and both the following two conditions are satisfied*:

(α) *if* $k\in S$ *and* $\mu(k)=\rho(w_1)\,\rho(w_2)^\sigma$ *or* $2\rho(w_2)\,\rho(w_2)^\sigma$, *then* $\mu(k+1)=\rho(w_1)\,\rho(w_2)^\sigma$ *or* $2\rho(w_1)\,\rho(w_1)^\sigma$.

(β) *if* $k\in S$ *and* $\mu(k)=\rho(w_2)\,\rho(w_1)^\sigma$ *or* $2\rho(w_1)\,\rho(w_1)^\sigma$, *then* $\mu(k+1)=\rho(w_2)\,\rho(w_1)^\sigma$ *or* $2\rho(w_2)\,\rho(w_2)^\sigma$.

Proof. The necessary part follows from Lemmas 4.13–4.15. It is easy to show that in case (i), (ii), or (iii), $l([\mu])=2m$. The converse part is very easy to check.

COROLLARY 4.17. $\langle\Gamma\eta_I,\rho_{I'}\rangle=1$ *for all* $I\in\Omega$.

COROLLARY 4.18. *For* $I\in\Omega$, *the following two statements are equivalent*:

(i) $\Gamma\eta_{I'}=\Phi_I$,

(ii) *there exists* $k\in S$ *such that one of the following holds*:

(α) $I(k)=(1,1)$,

(β) $I(k)=(1,0)$ *and* $I(k+1)=(0,1)$,

(γ) $I(k)=(0,1)$ *and* $I(k+1)=(1,0)$.

Proof. By Brauer's orthogonality relations and Corollary 4.17, we have

$$\Gamma\eta_{I'}=\Phi_I+\sum_{I\neq J\in\Omega}\langle\Gamma\eta_{I'},\rho_J\rangle\,\Phi_J.$$

Thus $\eta_{I'}=\Phi_I$ if and only if $\langle\Gamma\eta_{I'},\rho_J\rangle=0$ for all $J\in\Omega, J\neq I$. Now the corollary follows from Lemma 4.11. We leave the routine verification as an easy exercise.

DEFINITION 4.19. *$I \in \Omega$ is said to be well behaved if I satisfies one of the two equivalent conditions in Corollary* 4.18. *Otherwise, I is said to be non well behaved.*

COROLLARY 4.20. *Suppose $I, J \in \Omega$, $I \neq J'$, and $\langle \Gamma\eta_I, \rho_I \rangle \neq 0$. Then* (i) *one of the following holds*:

(α) *there exists $k \in S$ such that $J(k) = (1, 1)$ and hence J is well behaved.*

(β) $I = T$, $J = (S, \phi)$,

(γ) $I = T$, $J = (\phi, S)$.

(ii) *one of the following holds*:

(α) *there exists $k \in S$ such that $I(k) = (1, 1)$ and hence I is well behaved.*

(β) $I = (S, \phi)$, $J = T$,

(γ) $I = (\phi, S)$, $J = T$.

Proof. Suppose $J(k) \neq (1, 1)$ for all $k \in S$. As $\langle \Gamma\eta_I, \rho_I \rangle \neq 0$, so there exist μ as in Proposition 4.16. By assumption, μ satisfies Proposition 4.16(iii). By Lemmas 4.11 4.12 and the assumption $I \neq J'$, we get that $J(k) \neq (0, 0)$ for all $k \in S$. Hence $J(k) = (1, 0)$ or $(0, 1)$ for all $k \in S$. By Lemma 4.11(ii) and (iii), $\mu(k) = \rho(w_1)\rho(w_2)^\sigma$ or $\rho(w_2)\rho(w_1)^\sigma$ for all $k \in S$. Suppose $\mu(j) = \rho(w_1)\rho(w_2)^\sigma$ for some $j \in S$. By Proposition 4.16(iii)(α), $\mu(k) = \rho(w_1)\rho(w_2)^\sigma$ for all $k \in S$. By Lemma 4.11(iii), $I(k) = (1, 1)$ and $J(k) = (1, 0)$. Therefore, $I = T$, $J = (S, \phi)$. Similarly, if $\mu(j) = \rho(w_2)\rho(w_1)^\sigma$ for some $j \in S$, then $I = T$, $J = (\phi, S)$. Condition (ii) can be proved in the same way.

COROLLARY 4.21. (i) $\langle \Gamma\eta_{(S,\phi)}, \Gamma \rangle = \langle \Gamma\eta_{(\phi,S)}, \Gamma \rangle = \langle \Gamma\eta_T, \rho_{(S,\phi)} \rangle = \langle \Gamma\eta_T, \rho_{(\phi,S)} \rangle = 1$.

(ii) $\Gamma\eta_{(S,\phi)} = \Phi_{(S,\phi)} + \Gamma$, $\Gamma\eta_{(\phi,S)} = \Phi_{(\phi,S)} + \Gamma$.

Proof. These are easy results of Proposition 4.16 and Brauer's orthogonality relations.

THEOREM 4.22. (i) *Suppose $I\in\Omega$ and $I\neq T$. Then*

$$\Phi_{I'}=\Gamma\eta_I-\sum_{I'\neq J\in\Omega}\langle\Gamma\eta_I,\rho_J\rangle\,\Gamma\eta_{J'}.$$

(ii) $\Phi_{T'}=\Gamma\eta_T+2\Gamma-\sum_{T'\neq J\in\Omega}\langle\Gamma\eta_T,\rho_J\rangle\,\Gamma\eta_{J'}$.

Proof. The proof of (i) follows from Corollaries 4.17, 4.18, and 4.20 and (ii) from Corollaries 4.17, 4.18, 4.20, and 4.21.

5. THE DEGREES OF PROJECTIVE INDECOMPOSABLE CHARACTERS

The formulas for the degrees of the principal indecomposable modules in characteristic 2 of the group G have already been obtained in [2]. Here we give another way to compute them.

For $I, J\in\Omega$, we define

$$\begin{aligned}(\Gamma\eta_I,\rho_J)&=\langle\Gamma\eta_T,\rho_T\rangle-1, &&\text{if}\quad I=J=T,\\ &=\langle\Gamma\eta_I,\rho_J\rangle, &&\text{otherwise.}\end{aligned}\tag{5.1}$$

The reason we replace $\langle,\rangle$ by $(,)$ is: if $\langle\Gamma\eta_I,\ \rho_J\rangle\neq 0$ and μ is as in Proposition 4.16 and $I\neq J'$, then $\langle\Gamma,\ [\mu]\rangle=2^t$, where

$$t=|\{k\in S\,|\,\mu(k)=2\rho(w_1)\,\rho(w_1)^\sigma\ \text{or}\ 2\rho(w_2)\,\rho(w_2)^\sigma\}|.$$

This means that each $k\in S$ with $\mu(k)=2\rho(w_1)\,\rho(w_1)^\sigma$ or $2\rho(w_2)\,\rho(w_2)^\sigma$ contributes 2 to $\langle\Gamma,\ [\mu]\rangle$. $(\Gamma\eta_I,\ \rho_J)$ is the sum of all $\langle\Gamma,\ [\mu]\rangle$ with all possible μ's satisfying Proposition 4.16(iii). That is, the 1 subtracted in (5.1) means that we neglect the contribution of the μ in Proposition 4.16(ii). By (5.1) and Theorem 4.22, we get easily

LEMMA 5.2. (i) *Suppose $I\in\Omega$ and $I\neq T$. Then*

$$\Phi_{I'}(1_G)=\Gamma(1_G)\left[\eta_I(1_G)-\sum_{I'\neq J\in\Omega}(\Gamma\eta_I,\rho_J)\,\eta_{J'}(1_G)\right].$$

(ii) $\Phi_{(\Phi,\Phi)}(1_G)=\Gamma(1_G)[\eta_T(1_G)+1-\sum_{T'\neq J\in\Omega}(\Gamma\eta_I,\rho_J)\,\eta_{J'}(1_G)]$.

Since we know that $\Gamma(1_G)=q^3$ and

$$\eta_I(1_G)=3^{N(I,(1,0))+N(I,(0,1))}\,6^{N(I,(1,1))}=3^{|I_1\cup I_2|}\,2^{|I_1\cap I_2|}$$

for all $I=(I_1,I_2)\in\Omega$, to compute $\Phi_{I'}(1_G)$ we need only consider the term

$$\sum_{I'\neq J\in\Omega}(\Gamma\eta_I,\rho_J)\,\eta_{J'}(1_G)\tag{5.3}$$

for both cases $I \neq T$ and $I = T$. In the rest of this section, $I = (I_1, I_2)$ is an arbitrary fixed element in Ω, unless otherwise stated.

By Lemmas 4.8 and 4.11. and Propostion 4.16(iii), we get that if $J \in \Omega$, $J \neq I'$ and $(\Gamma\eta_I, \rho_J) \neq 0$, then

(i) $I(k) \neq (0, 0)$ for all $k \in S$,

(ii) if $I(k) = (1, 0)$ or $(0, 1)$ for some $k \in S$, then $J(k) = (1, 1)$,

(iii) if $I(k) = (1, 1)$ for some $k \in S$, then $J(k) \neq (0, 0)$.

These results lead to the following notation: For a subset A of S, we denote $\mathfrak{F}(I, A)$ as the set of all $J(A) = (J(A)_1, J(A)_2) \in \Omega(A)$ such that

(i) if $I(k) = (1, 0)$ or $(0, 1)$ for some $k \in A$, then $J(A)(k) = (1, 1)$,

(ii) if $I(k) = (1, 1)$ for some $k \in A$, then $J(A)(k) \neq (0, 0)$.

Define $\mathfrak{F}(I) = \mathfrak{F}(I, S)$. Clearly, $\{J \in \Omega \mid J \neq I',\ (\Gamma\eta_I, \rho_J) \neq 0\} \subset \mathfrak{F}(I)$. For $J(A) \in \mathfrak{F}(I, A)$, define $\mathfrak{H}(I, J(A), A)$ as the set of all functions $\mu: A \to \{\rho(w_1)\rho(w_2)^\sigma,\ \rho(w_2)\rho(w_1)^\sigma,\ 2\rho(w_1)\rho(w_1)^\sigma,\ 2\rho(w_2)\rho(w_2)^\sigma\}$ such that for $k \in A$, $\mu(k)$ is a term in the formula for

$$\overline{\eta(I_1(k)\, w_1 + I_2(k)\, w_2)}\, \rho(J(A)_1\,(k)\, w_1 + J(A)_2\,(k)\, w_2)$$

as stated in Lemma 4.8 and

(i) if $k,\ k+1 \in A$ and $\mu(k) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$, then $\mu(k+1) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$,

(ii) if $k,\ k+1 \in A$ and $\mu(k) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$, then $\mu(k+1) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$.

We note that it may happen $\mathfrak{H}(I, J(A), A) = \varnothing$. Define $\mathfrak{H}(I, J(S)) = \mathfrak{H}(I, J(A), A)$. Then there is a one-to-one correspondence between the set $\mathfrak{H}(I, J(S))$ and the set of μ's defined in Proposition 4.16(iii) with $J = J(S)$. For $k \in A$, $\mu \in \mathfrak{H}(I, J(A), A)$, define

$$\begin{aligned} c(\mu, k) &= 1, && \text{if } I(k) = (0,0) \text{ or } (0, 1), \\ &= 2, && \text{if } I(k) = J(A)(k) = (1, 1), \\ &= 3, && \text{if } I(k) = (1, 1) \text{ and } J(A)(k) = (1, 0) \text{ or } (0, 1). \end{aligned}$$

In fact, $c(\mu, k)$ is the product of two numbers: the coefficient of $\mu(k)$ and the number 1 (resp. 3), which depends on $J(A)(k) = (1, 1)$ (resp. $(0, 1)$ or $(1, 0)$). The latter number is the contribution of k to $\eta_{J'}(1_G)$ in $(\Gamma\eta_I,\ \rho_J)\,\eta_{J'}(1_G)$. Using the above notation, we can get easily that (5.3) is equal to

$$\sum_{J \in \mathfrak{F}(I)} \sum_{\mu \in \mathfrak{H}(I,J)} \prod_{k \in S} c(\mu, k). \tag{5.4}$$

Suppose $I(k) = (1, 1)$ for r consecutive k's in S. For convenience at this moment, suppose $I(k) = (1, 1)$ for $r = 0, 1, ..., r-1$. For $0 \leqslant t \leqslant r-1$, let $A_t = \{0, 1, ..., t-1\} \subset S$. Define

$$P_t = \sum_{J(A_t) \in \mathfrak{F}(I, A_t)} \sum_{\mu} \prod_{k \in A_t} c(\mu, k), \tag{5.5}$$

where the second summation ranges over all $\mu \in \mathfrak{H}(I, J(A_t), A_t)$ with $\mu(0) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$ and $\mu(t-1) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$. In other words, this summation ranges over all $\mu \in \mathfrak{H}(I, J(A_t), A_t)$ such that the term having length $2t$ in

$$[\mu] = \prod_{k \in A_t} \mu(k)^{\sigma^k}$$

is of the form

$$x\rho(w_1)\left(\prod_{k=1}^{t-1} \rho(w_1 + w_2)^{\sigma^k}\right)\rho(w_1)^{\sigma^t},$$

where x is a constant. Similarly, Q_t is defined in the same way as P_t, except $\mu(t-1) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$. In the same way, for P'_t, we require $\mu(0) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$ and $\mu(t-1) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_1)\rho(w_1)^\sigma$. For Q'_t, we require $\mu(0) = \rho(w_2)\rho(w_1)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$ and $\mu(t-1) = \rho(w_1)\rho(w_2)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$. Now, it is easy to see that if $1 \leqslant t \leqslant r-1$, then

$$P_{t+1} = 3P_t + 2Q_t, \qquad Q_{t+1} = 2P_t + 3Q_t$$

and

$$P'_{t+1} = 3P'_t + 2Q'_t, \qquad Q'_{t+1} = 2P'_t + 3Q'_t.$$

As $P_1 = 2, Q_1 = 3, P'_1 = 3, Q'_1 = 2$, we get

$$P_t = \tfrac{1}{2}(5^t - 1), \qquad Q_t = \tfrac{1}{2}(5^t + 1)$$

and (5.6)

$$P'_t = \tfrac{1}{2}(5^t + 1), \qquad Q'_t = \tfrac{1}{2}(5^t - 1)$$

for all $t = 1, 2, ..., r$.

If I' is well behaved, then by Corollary 4.18(i),

$$\Phi_{I'}(1_G) = \Gamma(1_G)\,\eta_I(1_G) = q^3\, 3^{|I_1 \cup I_2|}\, 2^{|I_1 \cap I_2|}.$$

In the rest of this section, we suppose that I' is non well behaved, i.e., I satisfies the following three conditions:

(i) $I(k) \neq (0,0)$ for all $k \in S$.

(ii) If $I(k) = (1,0)$ for some $k \in S$, then $I(k+1) = (1,0)$ or $(1,1)$.

(iii) If $I(k) = (0,1)$ for some $k \in S$, then $I(k+1) = (0,1)$ or $(1,1)$.

Now, suppose $I \neq T$ and let $I_1 \cap I_2 = \bigcup_{s=1}^{e} B_s$, where

(i) $B_s = \{i_s, i_s+1, i_s+2, \dots, j_s\}$ for some $i_s, j_s \in S$,

(ii) $i_s - 1, j_s + 1 \notin I_1 \cap I_2$,

(iii) $i_1, i_2, \dots, i_e$ are distinct.

Define

$$\begin{aligned} \varepsilon_s &= 0, \qquad \text{if} \quad I(i_s - 1) = I(j_s + 1), \\ &= 1, \qquad \text{if} \quad I(i_s - 1) \neq I(j_s + 1). \end{aligned}$$

Let $|B_s| = b_s$. I is said to be of type $\{(b_s, \varepsilon_s) | 1 \leqslant s \leqslant e\}$. Under this definition and the consideration of (5.6), we get that (5.3) and (5.4) are equal to

$$\prod_{s=1}^{e} \left(\frac{5^{b_s} + (-1)^{\varepsilon_s}}{2} \right). \tag{5.7}$$

Now we consider the case $I = T$. Equations (5.3) and (5.4) are actually equal to $Q_m + P'_m = 5^m + 1$. In summary, by Lemma 5.2, Eqs. (5.3), (5.4) and (5.7), we get

THEOREM 5.8 ([12, Theorem A, (7.6), Corollary, Theorem 8.2]). (i) $\Phi_{(\phi,\phi)}(1_G) = 2^{3m}(6^m - 5^m)$.

(ii) *If* $I = (I_1, I_2) \in \Omega$ *is well behaved, then*

$$\Phi_I(1_G) = 2^{3m + |I'_1 \cap I'_2|}\, 3^{|I'_1 \cup I'_2|}.$$

(iii) *If* $I = (I_1, I_2) \in \Omega$ *is non well behaved and* $I' \neq T$ *is of type* $\{(b_s, \varepsilon_s) | 1 \leqslant s \leqslant e\}$, *then*

$$\Phi_I(1_G) = 2^{3m} \left\{ 2^{|I'_1 \cap I'_2|} 3^{|I'_1 \cup I'_2|} - \prod_{s=1}^{e} \left(\frac{5^{b_s} + (-1)^{\varepsilon_s}}{2} \right) \right\}.$$

6. PRELIMINARY COMPUTATION OF $C_{\phi,\phi}$

The next three sections are devoted to computing $C_{\phi,\phi} = \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)} \rangle$. By Theorem 4.22 and (5.1),

$$\Phi_{(\phi,\phi)} = \Gamma\eta_T + \Gamma - \sum_{(\phi,\phi) \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\, \Gamma\eta_{I'}.$$

As it is well known that $\langle \Gamma, \Phi_{(\phi,\phi)}\rangle = 0$, so

$$\begin{aligned}
C_{\phi,\phi} &= \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)}\rangle \\
&= \langle \Gamma\eta_T, \Phi_{(\phi,\phi)}\rangle - \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Phi_{(\phi,\phi)}\rangle \\
&= \langle \Gamma\eta_T, \Gamma\eta_T\rangle + \langle \Gamma\eta_T, \Gamma\rangle - \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\rangle \\
&\quad - \sum_{(\phi,\phi)\neq J\in\Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle \\
&\quad - \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle \\
&\quad + \sum_{(\phi,\phi)\neq I,J\in\Omega} (\Gamma\eta_T, \rho_I)(\Gamma\eta_T, \rho_J)\langle \Gamma\eta_{I'}, \Gamma\eta_{J'}\rangle. \qquad (6.1)
\end{aligned}$$

We first consider the term $\sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\rangle$.

LEMMA 6.2. *Let $I\in\Omega$ and $I\neq(\phi,\phi)$. Then*

$$\begin{aligned}
(\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\rangle &= \langle \Gamma\eta_T, \Gamma\rangle - 1, && \text{if } I = T, \\
&= 1, && \text{if } I = (S,\phi) \text{ or } (\phi, S), \\
&= 0, && \text{otherwise.}
\end{aligned}$$

Proof. The first case is trivial. The second one comes from Corollary 4.21(i). Now, we suppose that $I\neq T$, $I\neq(S,\phi)$ and $I\neq(\phi,S)$. If $\langle \Gamma\eta_{I'}, \Gamma\rangle\neq 0$, then, by Corollary 4.20(ii), there exists $k\in S$ such that $I(k)=(0,0)$; $(\Gamma\eta_T, \rho_I) = \langle \Gamma\eta_T, \rho_I\rangle$ as $I\neq T$. By Proposition 4.16(i) and Lemma 4.15(i), we get that $(\Gamma\eta_T, \rho_I) = 0$. This completes the proof.

Lemma 6.2 gives

$$\langle \Gamma\eta_T, \Gamma\rangle - \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_I', \Gamma\rangle = -1. \qquad (6.3)$$

Before computing the other terms in (6.1), we introduce some notation and terminology. We remark that the notation and the method we will use to compute the other terms in (6.1) can also be applied to Section 4 to get the main results, Proposition 4.16. But instead of using these, we used the length function there.

For $n\geqslant m$, the set $\mathfrak{B}_m = \{\rho_I | I\in\Omega=\Omega(m)\}$ can be naturally identified as a subset of $\mathfrak{B}_n = \{\rho_I | I\in\Omega(n)\}$, the set of all irreducible Brauer characters of $G(n)$. In the following, we require that n is much greater than m (in fact, $n\geqslant m+4$ is enough), and that all the computation is within $R(G(n))$, unless otherwise stated, e.g., inner product is always considered as within $R(G(m))=R(G)$.

The following notation will be useful to simplify the notation and the computation. For r, s, t, $u \in \{0, 1\} \subset Z$, we use

$$\begin{vmatrix} r & t \\ s & u \end{vmatrix}$$

to denote the term $\rho(rw_1 + sw_2)\,\rho(tw_1 + uw_2)^\sigma$. For example,

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = \rho(0), \qquad \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = \rho(w_1), \qquad \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = \rho(w_1 + w_2)^\sigma, \ldots$$

Of course, we can generalize these to cover all elements in $\mathfrak{B}_n$. But the above is enough for our need. We remark that it is more effective to leave the "0" a blank space in the above notation. We do not use the latter notation.

DEFINITION 6.4. *Let $\mathfrak{A} \subset \mathfrak{B}_n$ and $r \in N$, $r \leqslant n$. We say that*

$$\rho_I = \prod_{i \in S(n)} \rho(I_1(i)\, w_1 + I_2(i)\, w_2)^{\sigma^i},$$

with $I = (I_1, I_2) \in \Omega(n)$, is of r pattern $\mathfrak{A}$ if

$$\prod_{i=0}^{r-1} \rho(I_1(i)\, w_1 + I_2(i)\, w_2)^{\sigma^i} \in \mathfrak{A}.$$

In general, if

$$\mu = \sum_{I \in \Omega(n)} a(\mu, I)\, \rho_I \in R(G(n)),$$

with $a(\mu, I) \in Z$ and ρ_I is of r pattern $\mathfrak{A}$ for every $I \in \Omega(n)$ with $a(\mu, I) \neq 0$, then we say that μ is of r pattern $\mathfrak{A}$. If $\mathfrak{A}$ consists of a single element ρ, we say, briefly, that μ is of r pattern ρ.

DEFINITION 6.5. *Let $\mathfrak{A} \subset \mathfrak{B}_n$ and $s \in N$, $s \leqslant n$. We say that*

$$\rho_I = \prod_{i \in S(n)} \rho(I_1(i)\, w_1 + I_2(i)\, w_2)^{\sigma^i},$$

with $I = (I_1, I_2) \in \Omega(n)$, is of s type $\mathfrak{A}$ if

$$\prod_{i=s}^{n-1} \rho(I_1(i)\, w_1 + I_2(i)\, w_2)^{\sigma^{i-s}} \in \mathfrak{A}.$$

In general, if

$$\mu = \sum_{I \in \Omega(n)} a(\mu, I)\, \rho_I \in R(G(n)),$$

with $a(\mu, I) \in Z$ and ρ_I is of s type $\mathfrak{A}$ for every $I \in \Omega(n)$ with $a(\mu, I) \neq 0$, then we say that μ is of s type $\mathfrak{A}$. If $\mathfrak{A}$ consists of a single element ρ, we say, briefly, that μ is of s type ρ.

DEFINITION 6.6. *Suppose r, $s \in N$ and $r \leqslant s \leqslant n$. We say that ρ_I, with $I = (I_1, I_2) \in \Omega(n)$, is (r, s) complete if r, $r+1, \ldots, s \in I_1 \cap I_2$.*

In the rest of this section, we will compute $\langle \Gamma \eta_T, \Gamma \eta_T \rangle = \langle \Gamma, \eta_T^2 \Gamma \rangle$ in (6.1). By a routine computation using Lemma 3.2, we get

$$\begin{aligned} &\eta(w_1 + w_2)^2\, \rho(w_1 + w_2) \\ &\quad = \rho(w_1 + w_2)\, \rho(w_1 + w_2)^\sigma + 4\rho(w_1 + w_2) + 2\rho(w_1)\, \rho(w_2)^{\sigma^2} \qquad (6.7) \\ &\qquad + 2\rho(w_2)\, \rho(w_1)^{\sigma^2} + 6\rho(w_1 + w_2)^\sigma + 6\rho(w_1)\, \rho(w_1)^\sigma + 6\rho(w_2)\, \rho(w_2)^\sigma. \end{aligned}$$

By (4.2),

$$\eta_T^2 \Gamma = \prod_{i=0}^{m-1} (\eta(w_1 + w_2)^2\, \rho(w_1 + w_2))^{\sigma^i}|_G.$$

For $A \subset \{0, 1, 2, \ldots, m-1\}$, let $\mathfrak{J}(A)$ be the set of all functions

$$\begin{aligned} \mu\colon A \to C \equiv \{&\rho(w_1 + w_2)\, \rho(w_1 + w_2)^\sigma, 4\rho(w_1 + w_2), 2\rho(w_1)\, \rho(w_2)^{\sigma^2}, \\ &2\rho(w_2)\, \rho(w_1)^{\sigma^2}, 6\rho(w_1 + w_2)^\sigma, \\ &6\rho(w_1)\, \rho(w_1)^\sigma, 6\rho(w_2)\, \rho(w_2)^\sigma\}. \end{aligned}$$

In case $A = S$, we write $\mathfrak{J} = \mathfrak{J}(S)$. Actually,

$$\eta_T^2 \Gamma = \sum_{\mu \in \mathfrak{J}} \prod_{i=0}^{m-1} \mu(i)^{\sigma^i}|_G. \qquad (6.8)$$

Define

$$\mathfrak{C} = \left\{ \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \right\}.$$

Lemmas 6.9–6.11 are derived from Lemma 3.2 and induction on k. We leave the proof to the reader.

LEMMA 6.9. *Let $\mu_1, \mu_2 \in C$. Then $\mu_1 \mu_2^\sigma$ is of 2 type $\mathfrak{C}$.*

LEMMA 6.10. *Let $\mu_1 \in \mathfrak{C}$ and $\mu \in \mathfrak{J}$. For $1 \leqslant k \leqslant m$, let*

$$\mu_1 \prod_{i=0}^{k-1} \mu(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(I)\, \rho_I,$$

with $a(I) \in Z$. If $a(I) \neq 0$ and ρ_I is $(1, k-1)$ complete, then ρ_I is of k type $\mathfrak{C}$.

LEMMA 6.11. *For any $\mu \in \mathfrak{J}$ and $1 \leqslant k \leqslant m$,*

$$\prod_{i=0}^{k-1} \mu(i)^{\sigma^i}$$

is of k type

$$\mathfrak{C} \cup \left\{\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}, \begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}, \begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}, \begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}, \rho(w_1)^{\sigma^2}, \rho(w_2)^{\sigma^2}\right\}.$$

LEMMA 6.12. *Let $\mu \in \mathfrak{J}$ and*

$$\prod_{i=0}^{m-1} \mu(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(I)\, \rho_I, \tag{6.13}$$

with $a(I) \in Z$. Suppose that for some $I \in \Omega(n)$, $a(I) \neq 0$ and ρ_I is of m type $\{\rho(w_1)^{\sigma^2}, \rho(w_2)^{\sigma^2}\}$. Then ρ_I, when restricted to $G = G(m)$, has length less than $2m$.

Proof. The case $m = 1$ is trivial. Since $\mu(0)\,\mu(1)^\sigma$ is of 2 type $\mathfrak{C}$ by Lemma 6.9, the result is true for $m = 2$. Hence, we may assume $m > 2$. Suppose ρ_I is of m type $\rho(w_1)^{\sigma^2}$, $a(I) \neq 0$, and $\rho_I|_G$ has length $2m$. Then by Lemma 6.11,

$$\rho_I = \rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma\,\rho(w_2)^{\sigma^2}\left(\prod_{i=3}^{m-1} \rho(w_1 + w_2)^{\sigma^i}\right)\rho(w_1)^{\sigma^{m+2}}.$$

This implies that $\mu(0) = 4\rho(w_1 + w_2)$ or $\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma$ and $\mu(1) = 4\rho(w_1 + w_2)$, $\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma$, or $6\rho(w_1 + w_2)^\sigma$. The terms of $\mu(0)\,\mu(1)^\sigma$ that contribute to ρ_I should be

$$\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma \qquad \text{or} \qquad \rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma\,\rho(w_1 + w_2)^{\sigma^2},$$

here we omit the coefficients. Let $\mu_1 = 1$ or $\rho(w_1 + w_2)$, and

$$\left(\mu_1 \prod_{i=2}^{m-1} \mu(i)^{\sigma^{i-2}}\right)^{\sigma^2} = \left(\sum_{J \in \Omega(n)} b(J)\,\rho_I\right)^{\sigma^2},$$

with $b(J) \in Z$. Suppose $b(J) \neq 0$ and ρ_I is of $(m-2)$ pattern

$$\rho(w_2) \prod_{i=1}^{m-3} \rho(w_1 + w_2)^{\sigma^{i-2}},$$

then, by Lemma 6.9, ρ_I is of $(m-2)$ type $\mathfrak{C}$. As ρ_I is a nontrivial term in

$$\rho(w_1 + w_2)\,\rho(w_1 + w_2)^{\sigma}\mu_1 \prod_{i=2}^{m-1} \mu(i)^{\sigma^i}$$

and ρ_I is of $(k-2)$ pattern

$$\rho(w_1 + w_2)\,\rho(w_1 + w_2)^{\sigma}\,\rho(w_2)^{\sigma^2} \prod_{i=3}^{m-3} \rho(w_1 + w_2)^{\sigma^i},$$

so ρ_I is of m type $\mathfrak{C}$. But $\rho(w_1)^{\sigma^2} \notin \mathfrak{C}$, a contradiction. This shows that ρ_I has length less than $2m$. For the case ρ_I is of m type $\rho(w_2)^{\sigma}$, we take the complex conjugate, i.e., the bar action, in (6.13). Then the result follows easily. This completes the proof.

LEMMA 6.14. *Let $\mu \in \mathfrak{J}$ and*

$$\prod_{i=0}^{m-1} \mu(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(I)\,\rho_I,$$

with $a(I) \in Z$. Suppose that $a(I) \neq 0$ and ρ_I, when restricted to $G = G(m)$, has length $2m$. Then ρ_I is $(2, m-1)$ complete and one of the following 11 cases holds:

	2 pattern of ρ_I	m type of ρ_I		2 pattern of ρ_I	m type of ρ_I
1	$\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix}$	$\begin{vmatrix}0 & 0\\0 & 0\end{vmatrix}$	7	$\begin{vmatrix}1 & 0\\1 & 1\end{vmatrix}$	$\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}$
2	$\begin{vmatrix}0 & 1\\0 & 1\end{vmatrix}$ or $\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix}$	$\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}$	8	$\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix}$	$\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$
3	$\begin{vmatrix}0 & 0\\1 & 1\end{vmatrix}$	$\begin{vmatrix}1 & 1\\0 & 0\end{vmatrix}$	9	$\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}$	$\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix}$
4	$\begin{vmatrix}1 & 1\\0 & 0\end{vmatrix}$	$\begin{vmatrix}0 & 0\\1 & 1\end{vmatrix}$	10	$\begin{vmatrix}1 & 1\\1 & 0\end{vmatrix}$	$\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}$
5	$\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}$ or $\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix}$	$\begin{vmatrix}0 & 1\\0 & 1\end{vmatrix}$	11	$\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$	$\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix}$
6	$\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix}$	$\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}$			(6.15)

Proof. By Lemmas 6.11 and 6.12, ρ_I is of m type

$$\mathfrak{C} \cup \left\{\begin{vmatrix}1&1\\1&0\end{vmatrix}, \begin{vmatrix}1&0\\1&1\end{vmatrix}, \begin{vmatrix}1&1\\0&1\end{vmatrix}, \begin{vmatrix}0&1\\1&1\end{vmatrix}\right\}.$$

As $\rho_I|_G$ has length $2m$, so ρ_I is $(2, m-1)$ complete. Lemma 6.9 says that $\mu(0)\,\mu(1)^\sigma$ is of 2 type $\mathfrak{C}$. By Lemma 6.10, ρ_I is of m type $\mathfrak{C}$. Now, tabulation (6.15) follows easily from the condition that $\rho_I|_G$ has length $2m$.

For $1 \leqslant t \leqslant m$, let $A_t = \{0, 1, 2,..., t-1\} \subset S(n)$. For $\mu \in \mathfrak{J}(A_t)$, let

$$\prod_{i=0}^{t-1} \mu(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(\mu, I)\,\rho_I,$$

with $a(\mu, I) \in Z$. We arrange the elements in $\mathfrak{C}$ as follows (cf. (6.15) and Lemma 6.14):

$$\mathfrak{C}(1) = \begin{vmatrix}0&0\\0&0\end{vmatrix}, \qquad \mathfrak{C}(2) = \begin{vmatrix}1&0\\1&0\end{vmatrix}, \qquad \mathfrak{C}(3) = \begin{vmatrix}1&1\\0&0\end{vmatrix},..., \mathfrak{C}(11) = \begin{vmatrix}0&1\\1&0\end{vmatrix}.$$

We also arrange 2 patterns of ρ_I in (6.15) as

$$\mathfrak{D}(1) = \begin{vmatrix}1&1\\1&1\end{vmatrix}, \qquad \mathfrak{D}(2) = \begin{vmatrix}0&1\\0&1\end{vmatrix}, \qquad \mathfrak{D}(3) = \begin{vmatrix}0&0\\1&1\end{vmatrix},..., \mathfrak{D}(11) = \begin{vmatrix}1&0\\0&1\end{vmatrix}.$$

For fixed $1 \leqslant i, j \leqslant 11$, and $1 \leqslant t \leqslant m$, let

$$P(\mathfrak{D}(i), \mathfrak{C}(j), t) = \sum a(\mu, I)\,\rho_I, \tag{6.16}$$

where the summation ranges over all $\mu \in \mathfrak{J}(A_t)$, $I \in \Omega(n)$ such that ρ_I is of 2 pattern $\mathfrak{D}(i)$ and of t type $\mathfrak{C}(j)$ and $(2, t-1)$ complete. By Lemma 6.14,

$$\left(\sum_{i=1}^{11} P(\mathfrak{D}(i), \mathfrak{C}(i), m)\right) + P(\mathfrak{D}(1), \mathfrak{C}(2), m) + P(\mathfrak{D}(1), \mathfrak{C}(5), m)$$

is the sum of terms in

$$\sum_{\mu \in \mathfrak{J}, I \in \Omega(n)} a(\mu, I)\,\rho_I$$

with length $2m$, when restricted to $G = G(m)$. If we write

$$p(\mathfrak{D}(i), \mathfrak{C}(j), t) = \sum a(\mu, I), \tag{6.17}$$

where the summation is the same as in (6.16), then by (6.8),

$$\langle \Gamma, \eta_T^2 \Gamma \rangle = \left(\sum_{i=1}^{11} p(\mathfrak{D}(i), \mathfrak{C}(i), m) \right) + 2p(\mathfrak{D}(1), \mathfrak{C}(2), m) + 2p(\mathfrak{D}(1), \mathfrak{C}(5), m). \tag{6.18}$$

Our next step is to express (6.17) in an explicit form. We will do it by solving some linear recursive equations.

LEMMA 6.19. *For a fixed* $1 \leqslant i \leqslant 11$, *let* p_t *be the column vector*

$$(p(\mathfrak{D}(i), \mathfrak{C}(j), t))_{j=1}^{11}$$

with $t = 2, 3, ..., m$. *Then* $p_{t+1} = X_1 p_t$, *where*

$$X_1 = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & V_1 & W_1 \\ 0 & W_1 & V_1 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 4 & 8 & 6 & 6 & 4 \\ 1 & 8 & 6 & 6 & 6 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 7 & 12 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

Moreover, $p_m = X_1^{m-2} p_2$.

Proof. For every $\xi \in \mathfrak{J}(A_{t+1})$, there exists exactly one $\mu \in \mathfrak{J}(A_t)$ such that $\xi(k) = \mu(k)$ for $k = 0, 1, ..., t-1$.

$$\prod_{i=0}^{t} \xi(i)^{\sigma^i} = \left(\sum_{I \in \Omega(n)} a(\mu, I)\, \rho_I \right) \xi(t)^{\sigma^t}.$$

By (6.16) or (6.17) and the definition of $(2, t)$ completeness, $\xi(t)$ should be chosen carefully so that it constructs a suitable $(2, t)$-complete element in $P(\mathfrak{D}(\mathrm{i}), \mathfrak{C}(j), t+1)$. For example, if we want to express

$$P\left(\mathfrak{D}(i), \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}, t+1\right)$$

in terms of $P(\mathfrak{D}(i), \mathrm{C}(j), t)$ with $j = 1, 2, ..., 11$, then $\xi(t)$ should be chosen in the following way:

For $P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 0 & 0\end{vmatrix}, t\right)$, choose $\xi(t) = \rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma$.

For $P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}, t\right)$, choose $\xi(t) = \rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma$ or $6\rho(w_1 + w_2)^\sigma$.

For $P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}, t\right)$, choose $\xi(t) = 6\rho(w_2)\,\rho(w_2)^\sigma$.

For $P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 1 & 1\end{vmatrix}, t\right)$, choose $\xi(t) = 6\rho(w_1)\,\rho(w_1)^\sigma$.

For $P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 1\\ 0 & 1\end{vmatrix}, t\right)$, choose $\xi(t) = \rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma$ or $4\rho(w_1 + w_2)$.

So,

$$
\begin{aligned}
&P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}, t+1\right)\\
&\quad = P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 0 & 0\end{vmatrix}, t\right)(\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma)^{\sigma^t}\\
&\qquad + P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}, t\right)(\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma + 6\rho(w_1 + w_2)^\sigma)^{\sigma^t}\\
&\qquad + P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}, t\right)(6\rho(w_2)\,\rho(w_2)^\sigma)^{\sigma^t}\\
&\qquad + P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 1 & 1\end{vmatrix}, t\right)(6\rho(w_1)\,\rho(w_1)^\sigma)^{\sigma^t}\\
&\qquad + P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 1\\ 0 & 1\end{vmatrix}, t\right)(\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma + 4\rho(w_1 + w_2))^{\sigma^t}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
p\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}, t+1\right) &= p\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 0 & 0\end{vmatrix}, t\right) + 8p\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}, t\right)\\
&\quad + 6p\left(\mathfrak{D}(i), \begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}, t\right) + 6p\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 1 & 1\end{vmatrix}, t\right)\\
&\quad + 6p\left(\mathfrak{D}(i), \begin{vmatrix}0 & 1\\ 0 & 1\end{vmatrix}, t\right).
\end{aligned}
$$

The second coefficient on the right-hand side comes from $2 + 6$, and the "2" comes from the formula $\rho(w_1 + w_2)^2 = 2\rho(w_1 + w_2) + \cdots$. The above example gives one method to construct the X_1 in this lemma. This gives the proof of the existence of such X_1. Another effective way to construct X_1 can be described as follows: We first write down an 11×11 matrix, give the appropriate name,

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}, \ldots, \text{etc.},$$

to the rows and columns. Then fit the matrix column by column. We take the column

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

as an example. Among the set

$$\begin{aligned} C = \{&\rho(w_1 + w_2)\,\rho(w_1 + w_2)^\sigma, 4\rho(w_1 + w_2), 2\rho(w_1)\,\rho(w_2)^\sigma, 2\rho(w_2)\,\rho(w_1)^\sigma, \\ &6\rho(w_1 + w_2)^\sigma, 6\rho(w_1)\,\rho(w_1)^\sigma, 6\rho(w_2)\,\rho(w_2)^\sigma\}, \end{aligned}$$

we have to choose $2\rho(w_2)\,\rho(w_1)^{\sigma^2}$ and $6\rho(w_2)\,\rho(w_2)^\sigma$ for $\xi(t)$.

$$\begin{aligned} &\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} (2\rho(w_2)\,\rho(w_1)^\sigma) \\ &\quad = (\rho(w_1)\,\rho(w_2)^\sigma)(2\rho(w_2)\,\rho(w_1)^{\sigma^2}) \\ &\quad = 2(\rho(w_1)\,\rho(w_2))(\rho(w_2)^\sigma\,\rho(w_1)^{\sigma^2}) \\ &\quad = 2(\rho(w_1 + w_2) + \cdots)(\rho(w_2)\,\rho(w_1)^\sigma)^\sigma. \end{aligned}$$

The meaning of this is that $\xi(t) = 2\rho(w_2)\,\rho(w_1)^{\sigma^2}$ gives a contribution

$$2P\left(\mathfrak{D}(i), \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, t\right) \quad \textit{to} \quad P\left(\mathfrak{D}(i), \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, t + 1\right).$$

Write down 2 in the entry

$$\left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right).$$

Similarly,

$$\begin{aligned} &\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} (6\rho(w_1)\,\rho(w_1)^\sigma) \\ &\quad = (\rho(w_1)\,\rho(w_2)^\sigma)(6\rho(w_2)\,\rho(w_2)^\sigma) = 6(\rho(w_1)\,\rho(w_2))(\rho(w_2)^2)^\sigma \\ &\quad = 6(\rho(w_1 + w_2) + \cdots)(\rho(w_2)^\sigma + 2\rho(w_1))^\sigma. \end{aligned}$$

The meaning of this is that $\xi(t) = 6\rho(w_2)\,\rho(w_2)^\sigma$ gives a contribution

$$6P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}, t\right) \qquad \text{to} \qquad P\left(\mathfrak{D}(i), \begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}, t+1\right)$$

and gives a contribution

$$12P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}, t\right) \qquad \text{to} \qquad P\left(\mathfrak{D}(i), \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}, t+1\right).$$

Therefore, write down 6 in the entry

$$\left(\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}, \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\right)$$

and 12 in the entry

$$\left(\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}, \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\right).$$

Then put 0 in the other entries in the column

$$\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}.$$

This completes this column. Sometimes, it may happen that an entry has two numbers or more. In this case, put the sum of them in that entry. The reason for this is quite obvious.

Next, we have to find initial conditions for the recursive equations in Lemma 6.19.

LEMMA 6.20. *Let X_0 be the matrix whose jth column is the column vector*

$$(p(\mathfrak{D}(j), \mathfrak{C}(i), 2))_{i=1}^{11}.$$

Then

$$X_0 = \begin{pmatrix} U_0' & 0 & 0\\ 0 & V_0 & W_0\\ 0 & W_0 & V_0 \end{pmatrix}, \qquad U_0' = \begin{pmatrix} 24 & 48 & 84 & 84 & 44\\ 12 & 48 & 66 & 66 & 58\\ 0 & 0 & 4 & 0 & 6\\ 0 & 0 & 0 & 4 & 6\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_0=\begin{pmatrix} 50 & 0 & 0 \\ 0 & 24 & 48 \\ 0 & 4 & 10 \end{pmatrix}, \qquad W_0=\begin{pmatrix} 0 & 68 & 138 \\ 16 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Proof. This is nothing but a tedious computation. In this computation, we have to expand the term

$$(\eta(w_1+w_2)^2\rho(w_1+w_2))(\eta(w_1+w_2)^2\rho(w_1+w_2))^\sigma$$

(cf. (6.7)). Check the 2 pattern and 2 type of each term and put the coefficient in the appropriate entry in X_0. We leave the proof to the reader and give a comment here. In the actual computation, we find that the generalized notation of

$$\begin{matrix} | & | \\ | & | \end{matrix}$$

is more useful. In this, we denote the corresponding coefficient above the notation

$$\begin{matrix} | & | \\ | & | \end{matrix}.$$

Translating the formula in Lemma 3.2 and (6.7) into this symbolic notation will make the computation easier.

Let Y_0 be the matrix whose entries are the same as X_0 except

(i) the second column vector of Y_0 is the sum of the second one of X_0 and twice the first one of X_0,

(ii) the fifth column vector of Y_0 is the sum of the fifth one of X_0 and twice the first one of X_0. That is

$$Y_0=\begin{pmatrix} U_0 & 0 & 0 \\ 0 & V_0 & W_0 \\ 0 & W_0 & V_0 \end{pmatrix}, \qquad U_0=\begin{pmatrix} 24 & 96 & 84 & 84 & 92 \\ 12 & 72 & 66 & 66 & 82 \\ 0 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_0=\begin{pmatrix} 50 & 0 & 0 \\ 0 & 24 & 48 \\ 0 & 4 & 10 \end{pmatrix}, \qquad W_0=\begin{pmatrix} 0 & 68 & 138 \\ 16 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Now, by (6.18), $\langle \Gamma, \eta_T^2 \Gamma\rangle = \mathrm{tr}(X_1^{m-2}Y_0)$.

A mysterious phenomenon for which we cannot give a conceptual explanation is $X_1^2 = Y_0$. An intrinsic proof of this will be helpful to avoid the tedious computation in Lemma 6.20. By the fact $X_1^2 = Y_0$, we get

$$\langle \Gamma, \eta_T^2 \Gamma \rangle = \operatorname{tr} X_1^m,$$

which is the sum of the mth powers of the eigenvalues of X_1. The eigenvalues of X_1 are a, b, 2, 2, 1, 2, c, d, -2, $-c$, $-d$, where a, b are two roots of $x^2 - 12x + 24 = 0$ and c, d are two roots of $x^2 - 8x - 8 = 0$. As a result, we get

PROPOSITION 6.21.

$$\begin{aligned}\langle \Gamma\eta_T, \Gamma\eta_T \rangle = {} & 1 + 2^{m+1} + a^m + b^m + [2^m + c^m + d^m] \\ & + [(-2)^m + (-c)^m + (-d)^m],\end{aligned}$$

where a, b are two roots of $x^2 - 12x + 24 = 0$ and c, d are two roots of $x^2 - 8x - 8 = 0$.

7. COMPUTATIONS OF $C_{\phi,\phi}$ CONTINUED

In this section, we will compute the term

$$\sum_{T' \neq I, J \in \Omega} (\Gamma\eta_T, \rho_I)(\Gamma\eta_T, \rho_J)\langle \Gamma\eta_{I'}, \Gamma\eta_{J'} \rangle \tag{7.1}$$

which is the last term on the right-hand side of (6.1). We remind the reader that all the computation is within $R(G(n))$ unless otherwise stated, e.g., the inner product is always considered as within $R(G(m)) = R(G)$. Suppose $(\phi, \phi) \neq I \in \Omega$ and $(\Gamma\eta_T, \rho_I) \neq 0$. By Lemma 4.13, $I(k) \neq (0, 0)$ for all $k \in S$. Let us restate the meaning of $(,)$ (see (5.1)) in the form we need. For $A \subset \{0, 1, 2, \ldots, m-1\}$, let $\mathfrak{L}(A)$ be the set of all functions

$$\mu: A \to \{\rho(w_1)\,\rho(w_2)^\sigma, \rho(w_2)\,\rho(w_1)^\sigma, 2\rho(w_1)\,\rho(w_1)^\sigma, 2\rho(w_2)\,\rho(w_2)^\sigma\}$$

such that (cf. Proposition 4.16(iii)):

(α) if k, $k+1 \in A$ and $\mu(k) = \rho(w_1)\,\rho(w_2)^\sigma$ or $2\rho(w_2)\,\rho(w_2)^\sigma$, then $\mu(k+1) = \rho(w_1)\,\rho(w_2)^\sigma$ or $2\rho(w_1)\,\rho(w_1)^\sigma$,

(β) if k, $k+1 \in A$ and $\mu(k) = \rho(w_2)\,\rho(w_1)^\sigma$ or $2\rho(w_1)\,\rho(w_1)^\sigma$, then $\mu(k+1) = \rho(w_2)\,\rho(w_1)^\sigma$ or $2\rho(w_2)\,\rho(w_2)^\sigma$.

For $\mu \in \mathfrak{L}(A)$, define

$$t(A)(\mu) = |\{k \in A \mid \mu(k) = 2\rho(w_1)\,\rho(w_1)^\sigma \text{ or } 2\rho(w_2)\,\rho(w_2)^\sigma\}|.$$

For $\mu \in \mathfrak{L} = \mathfrak{L}(S)$, define

$$[\mu] = \prod_{i=0}^{m-1} \mu(i)^{\sigma^i}|_G.$$

Then $\langle \Gamma, [\mu] \rangle = 2^{t(\mu)}$, where $t(\mu) = t(S)(\mu)$. This means that each $k \in S$ with $\mu(k) = 2\rho(w_1)\rho(w_1)^\sigma$ or $2\rho(w_2)\rho(w_2)^\sigma$ contributes 2 to $\langle \Gamma, [\mu] \rangle$. For $\mu \in \mathfrak{L}(A)$ define $I(\mu) \in \Omega(A)$ as follows: for $k \in A$, let

$$\begin{aligned} I(\mu)(k) &= (1, 0), \qquad \text{if} \quad \mu(k) = \rho(w_2)\rho(w_1)^\sigma, \\ &= (0, 1), \qquad \text{if} \quad \mu(k) = \rho(w_1)\rho(w_2)^\sigma, \\ &= (1, 1), \qquad \text{if} \quad \mu(k) = 2\rho(w_1)\rho(w_1)^\sigma \quad \text{or} \quad 2\rho(w_2)\rho(w_2)^\sigma. \end{aligned}$$

According to this notation,

$$(\Gamma\eta_T, \rho_I) = \sum_{\substack{\mu \in \mathfrak{L} \\ I(\mu) = I}} \langle \Gamma, [\mu] \rangle. \tag{7.2}$$

We note that if $(\Gamma\eta_T, \rho_I) = 0$, then the summation on the right-hand side ranges over the empty set. By definition, it is 0. Now, (7.1) is equal to

$$\begin{aligned} &\sum_{\mu, \xi \in \mathfrak{L}} \langle \Gamma, [\mu] \rangle \langle \Gamma, [\xi] \rangle \langle \Gamma\eta_{I(\mu)'}, \Gamma\eta_{I(\xi)'} \rangle \\ &\quad = \sum_{\mu, \xi \in \mathfrak{L}} 2^{t(\mu) + t(\xi)} \langle \Gamma\eta_{I(\mu)'}, \Gamma\eta_{I(\xi)'} \rangle \\ &\quad = \sum_{\mu, \xi \in \mathfrak{L}} \langle \Gamma, (2^{t(\mu)}\eta_{\overline{I(\mu)'}})(2^{t(\xi)}\eta_{I(\xi)'})\, \Gamma \rangle \\ &\quad = \langle \Gamma, \sum_{\mu, \xi \in \mathfrak{L}} (2^{t(\mu)}\eta_{\overline{I(\mu)'}})(2^{t(\xi)}\eta_{I(\xi)'})\, \Gamma \rangle. \end{aligned} \tag{7.3}$$

By definition,

$$\begin{aligned} &(2^{t(\mu)}\eta_{\overline{I(\mu)'}})(2^{t(\xi)}\eta_{I(\xi)'})\, \Gamma \\ &\quad = 2^{t(\mu) + t(\xi)} \cdot \prod_{i=0}^{m-1} (\eta(I(\mu)_2'(i)\, w_1 + I(\mu)_1'(i)\, w_2)\, \eta(I(\xi)_1'(i)\, w_1 \\ &\qquad + I(\xi)_2'(i)\, w_2)\, \rho(w_1 + w_2))^{\sigma^i}. \end{aligned} \tag{7.4}$$

By a routine computation using Lemma 3.2, we get

$$\begin{aligned} \eta(w_1)^2 \rho(w_1 + w_2) = \rho(w_1 + w_2)\rho(w_1)^\sigma &+ 2\rho(w_1)\rho(w_2)^\sigma \\ &+ 4\rho(w_1)^\sigma + 6\rho(w_2), \end{aligned} \tag{7.5.1}$$

$$\eta(w_1)\,\eta(w_2)\,\rho(w_1+w_2) = 3\rho(w_1+w_2) + \rho(w_1+w_2)^\sigma + 2\rho(w_1)\,\rho(w_1)^\sigma + 2\rho(w_2)\,\rho(w_2)^\sigma + 4, \tag{7.5.2}$$

$$\eta(w_2)^2\,\rho(w_1+w_2) = \rho(w_1+w_2)\,\rho(w_2)^\sigma + 2\rho(w_2)\,\rho(w_1)^\sigma + 4\rho(w_2)^\sigma + 6\rho(w_1), \tag{7.5.3}$$

$$2\eta(0)\,\eta(w_1)\,\rho(w_1+w_2) = 2\rho(w_2)\,\rho(w_1)^\sigma + 4\rho(w_2)^\sigma + 6\rho(w_1), \tag{7.5.4}$$

$$2\eta(0)\,\eta(w_2)\,\rho(w_1+w_2) = 2\rho(w_1)\,\rho(w_2)^\sigma + 4\rho(w_1)^\sigma + 6\rho(w_2), \tag{7.5.5}$$

$$4\eta(0)\,\rho(w_1+w_2) = 4\rho(w_1+w_2). \tag{7.5.6}$$

For $A \subset \{0, 1, 2, \ldots, m-1\} \subset S(n)$ and $\mu, \xi \in \mathfrak{L}(A)$, let $\mathfrak{M}(\mu, \xi)$ be the set of all functions $\lambda\colon A \to E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, where E_i is the set of terms in (7.5.i) for $1 \leqslant i \leqslant 6$ (e.g., $E_3 = \{\rho(w_1+w_2)\,\rho(w_2)^\sigma,\ 2\rho(w_2)\,\rho(w_1)^\sigma,\ 4\rho(w_2)^\sigma,\ 6\rho(w_1)\}$) and for all $k \in A$, $\lambda(k)$ satisfies the condition

$$\begin{aligned}
\lambda(k) &\in E_1, && \text{if } I(\mu)(k) = (0,1), && I(\xi)(k) = (1,0),\\
&\in E_2, && \text{if } I(\mu)(k) = (1,0), && I(\xi)(k) = (1,0),\\
& && \text{or } I(\mu)(k) = (0,1), && I(\xi)(k) = (0,1),\\
&\in E_3, && \text{if } I(\mu)(k) = (1,0), && I(\xi)(k) = (0,1),\\
&\in E_4, && \text{if } I(\mu)(k) = (0,1), && I(\xi)(k) = (1,1),\\
& && \text{or } I(\mu)(k) = (1,1), && I(\xi)(k) = (1,0),\\
&\in E_5, && \text{if } I(\mu)(k) = (1,0), && I(\xi)(k) = (1,1),\\
& && \text{or } I(\mu)(k) = (1,1), && I(\xi)(k) = (0,1),\\
&\in E_6, && \text{if } I(\mu)(k) = (1,1), && I(\xi)(k) = (1,1).
\end{aligned} \tag{7.6}$$

For $\mu, \xi \in \mathfrak{L} = \mathfrak{L}(S)$, we can get easily from this definition that

$$(2^{t(\mu)}\eta_{\overline{I(\mu)'}})(2^{t(\xi)}\eta_{I(\xi)'})\,\Gamma = \sum_{\lambda \in \mathfrak{M}(\mu,\xi)} \prod_{i=0}^{m=1} \lambda(i)^{\sigma^i}\Big|_G. \tag{7.7}$$

Let

$$\mathfrak{C} = \left\{\begin{vmatrix}0&0\\0&0\end{vmatrix}, \begin{vmatrix}1&0\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\1&0\end{vmatrix}, \begin{vmatrix}1&0\\1&0\end{vmatrix}, \begin{vmatrix}0&1\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\0&1\end{vmatrix}\right\}.$$

Lemmas 7.8–7.10 are derived from Lemma 3.2 and induction on k. We leave the proof to the reader.

LEMMA 7.8. *Let $\lambda \in E$. Then λ, $\rho(w_1)\lambda, \rho(w_2)\lambda$ are of* 1 *type* $\mathfrak{E}$ *and* $\rho(w_1+w_2)\lambda$ *is of* 1 *type*

$$\mathfrak{E} \cup \left\{\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}, \begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}\right\}.$$

LEMMA 7.9. *Suppose* $1 \leqslant k \leqslant m$ *and* $\lambda(0), \lambda(1),..., \lambda(k-1) \in E$. *Then*

$$\prod_{i=0}^{k-1} \lambda(i)^{\sigma^i}$$

is of k type

$$\mathfrak{E} \cup \left\{\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}, \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}, \begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}, \begin{vmatrix}0 & 0\\ 1 & 1\end{vmatrix}\right\}.$$

LEMMA 7.10. *Let*

$$\lambda \in \mathfrak{E} \cup \left\{\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}, \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\right\}.$$

Suppose $3 \leqslant k \leqslant m, \lambda(2), \lambda(3),..., \lambda(k-1) \in \mathfrak{E}$ *and*

$$\lambda^{\sigma^2} \prod_{i=2}^{k-1} \lambda(i)^{\sigma^i} = \sum_{I \in \Omega(n)} b(I)\rho_I,$$

with $b(I) \in Z$. *If* $b(I) \neq 0$ *and* ρ_I *is* $(2, k-1)$ *complete, then* ρ_I *is of k type E.*

LEMMA 7.11. *Let* $1 \leqslant k \leqslant m$, $\lambda(0), \lambda(1),..., \lambda(k-1) \in \mathfrak{E}$ *and*

$$\prod_{i=0}^{k-1} \lambda(i)^{\sigma^i} = \sum_{I \in \Omega(n)} b(I)\rho_I,$$

with $b(I) \in Z$. *If* $b(I) \neq 0$ *and* ρ_I *is* $(2, k-1))$ *complete, then* ρ_I *is of k type* $\mathfrak{E}$.

Proof. As $\lambda(0)$ is of 1 type $\{1, \rho(w_1), \rho(w_2), \rho(w_1+w_2)\}$, so $\lambda(0)\lambda(1)^{\sigma}$ is of 2 type

$$\mathfrak{E} \cup \left\{\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}, \begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}\right\}$$

by Lemma 7.8. By Lemma 7.10, ρ_I is of k type $\mathfrak{E}$.

Let $\lambda(0), \lambda(1),..., \lambda(m-1) \in E$ and

$$\prod_{i=0}^{m-1} \lambda(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(I)\rho_I,$$

with $a(I) \in Z$. Suppose $a(I) \neq 0$ and $\rho_I|_G$ has length $2m$. Then ρ_I is-$(2, m-1)$ complete by Lemma 7.9. Lemma 7.10 shows that ρ_I is of m type $\mathfrak{C}$. This implies that the 2 pattern of ρ_I is

$$\mathfrak{K} = \left\{ \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \right\}.$$

For $2 \leqslant k \leqslant m$, let

$$\prod_{i=0}^{k-1} \lambda(i)^{\sigma^i} = \sum_{J \in \Omega(n)} b(J, k)\, \rho_I,$$

with $b(J, k) \in Z$. There exists $J^k \in \Omega(n)$ with $b(J^k, k) \neq 0$ so that $x\rho_I$ is a nontrivial term of

$$\rho_{J^k} \prod_{i=k}^{m-1} \lambda(i)^{\sigma^i},$$

where x is a constant. Of course, the J^k is not unique. Since ρ_I is $(2, m-1)$ complete, so ρ_{J^k} is $(2, k-1)$ complete. Also, ρ_{J^k} is of 2 pattern $\mathfrak{K}$. Once again, by some routine computations and inducion on k, we can get Lemma 7.12. The detail is left to the reader.

LEMMA 7.12. *The notation is as above. Then*

(i) *if* ρ_{J^k} *is of* k *type* $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$, *then* $\lambda(i-1) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$ *and* ρ_{J^i} *is of* i *type* $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$ *for all* $1 \leqslant i \leqslant k$.

(ii) *if* ρ_{J^k} *is of* k *type* $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$, *then* $\lambda(i-1) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ *and* ρ_{J^i} *is of* i *type* $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$ *for all* $1 \leqslant i \leqslant k$.

(iii) *if* ρ_{J^k} *is of* k *type* $\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$, *then* $\lambda(i-1) = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$ *and* ρ_{J^i} *is of* i *type* $\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$ *for all* $1 \leqslant i \leqslant k$.

In summary, we get that ρ_I is $(2, m-1)$ complete and one of the following 6 cases holds:

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	2 pattern of ρ_I	m type of ρ_I		2 pattern of ρ_I	m type of ρ_I
1	$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$	4	$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$
2	$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$	5	$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$
3	$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$	6	$\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$

(7.13)

This also shows that except cases 4–6, which are discussed in Lemma 7.12, ρ_I is $(1, m-1)$ complete and one of the following three cases holds:

	1 pattern of ρ_I	m type of ρ_I
1	$\rho(w_1 + w_2)$	1
2	$\rho(w_1)$	$\rho(w_2)$
3	$\rho(w_2)$	$\rho(w_1)$

Now, by (7.3) and (7.7) we get that (7.1) is equal to

$$\left\langle \Gamma, \sum_{\mu, \xi \in \mathfrak{L}} \sum_{\lambda \in \mathfrak{M}(\mu, \xi)} \prod_{i=0}^{m-1} \lambda(i)^{\sigma^i} \right\rangle. \tag{7.14}$$

To compute (7.14), we have to study those terms in

$$\sum_{\mu, \xi \in \mathfrak{L}} \sum_{\lambda \in \mathfrak{M}(\mu, \xi)} \prod_{i=0}^{m-1} \lambda(i)^{\sigma^i} \tag{7.15}$$

whose restriction to G has length $2m$. For $1 \leqslant t \leqslant m$, let $A_t = \{0, 1, 2, ..., t-1\} \subset S(n)$. For $\mu, \xi \in \mathfrak{L}(A_t)$ and $\lambda \in \mathfrak{M}(\mu, \xi)$, let

$$\prod_{i=0}^{t-1} \lambda(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(\mu, \xi, \lambda, I)\, \rho_I$$

with $a(\mu, \xi, \lambda, I) \in Z$. Define (cf. Lemma 7.20)

$\mathfrak{K}(1) = (\rho(w_2), \rho(w_2), \rho(w_2)),$ $\quad \mathfrak{E}(1) = (\rho(w_1), \rho(w_1), \rho(w_1)),$

$\mathfrak{K}(2) = (\rho(w_2), \rho(w_2), \rho(w_1)),$ $\quad \mathfrak{E}(2) = (\rho(w_1), \rho(w_1), \rho(w_2)),$

$\mathfrak{K}(3) = (\rho(w_2), \rho(w_1), \rho(w_1 + w_2)),$ $\quad \mathfrak{E}(3) = (\rho(w_1), \rho(w_2), 1),$

$$\begin{aligned}
\mathfrak{K}(4) &= (\rho(w_2), \rho(w_1), \rho(w_1)), & \mathfrak{E}(4) &= (\rho(w_1), \rho(w_2), \rho(w_2)),\\
\mathfrak{K}(5) &= (\rho(w_1), \rho(w_1), \rho(w_1)), & \mathfrak{E}(5) &= (\rho(w_2), \rho(w_2), \rho(w_2)),\\
\mathfrak{K}(6) &= (\rho(w_1), \rho(w_1), \rho(w_2)), & \mathfrak{E}(6) &= (\rho(w_2), \rho(w_2), \rho(w_1)),\\
\mathfrak{K}(7) &= (\rho(w_1), \rho(w_2), \rho(w_1+w_2)), & \mathfrak{E}(7) &= (\rho(w_2), \rho(w_1), 1),\\
\mathfrak{K}(8) &= (\rho(w_1), \rho(w_2), \rho(w_2)), & \mathfrak{E}(8) &= (\rho(w_2), \rho(w_1), \rho(w_1))\\
\mathfrak{K}(9) &= (\rho(w_2), \rho(w_1), \rho(w_2)), & \mathfrak{E}(9) &= (\rho(w_1), \rho(w_2), \rho(w_1)),\\
\mathfrak{K}(10) &= (\rho(w_2), \rho(w_2), \rho(w_1+w_2)), & \mathfrak{E}(10) &= (\rho(w_1), \rho(w_1), 1),\\
\mathfrak{K}(11) &= (\rho(w_1), \rho(w_2), \rho(w_1)), & \mathfrak{E}(11) &= (\rho(w_2), \rho(w_1), \rho(w_2)),\\
\mathfrak{K}(12) &= (\rho(w_1), \rho(w_1), \rho(w_1+w_2)), & \mathfrak{E}(12) &= (\rho(w_2), \rho(w_2), 1).
\end{aligned}$$

We also write $\mathfrak{K}(i) = (\mathfrak{K}(i)_1, \mathfrak{K}(i)_2, \mathfrak{K}(i)_3)$ and $\mathfrak{E}(i) = (\mathfrak{E}(i)_1, \mathfrak{E}(i)_2, \mathfrak{E}(i)_3)$ for $1 \leqslant i \leqslant 12$. For fixed $1 \leqslant i, j \leqslant 12$ and $0 \leqslant t \leqslant m$, let

$$Q(\mathfrak{K}(i), \mathfrak{E}(j), t) = \sum a(\mu, \xi, \lambda, I)\, \rho_I, \tag{7.16}$$

where the summation ranges over all μ, $\xi \in \mathfrak{L}(A_t)$, $\lambda \in \mathfrak{M}(\mu, \xi)$ such that

(i) the only term of length $2t$ (considered as in $G(n)$) in

$$[\mu] = \prod_{i=0}^{t-1} \mu(i)^{\sigma^i}$$

is of 1 pattern $\mathfrak{K}(i)_1$ and is of t type $\mathfrak{E}(i)_1$,

(ii) the only term of length $2t$ (considered as in $G(n)$) in

$$[\xi] = \prod_{i=0}^{t-1} \xi(i)^{\sigma^i}$$

is of 1 pattern $\mathfrak{K}(i)_2$ and is of t type $\mathfrak{E}(i)_2$,

(iii) ρ_I is of 1 pattern $\mathfrak{K}(i)_3$, of t type $\mathfrak{E}(i)_3$ and is $(1, t-1)$ complete.

By the above discussion, we get that

$$\begin{aligned}
&\sum_{i=1}^{12} Q(\mathfrak{K}(i), \mathfrak{E}(i), m) + \rho(w_1+w_2)\,\rho(w_2)^{\sigma} \left(\prod_{i=2}^{m-1} \rho(w_1+w_2)^{\sigma^i}\right) \rho(w_1)^{\sigma^{m+1}}\\
&\quad + \rho(w_1+w_2)\,\rho(w_1)^{\sigma} \left(\prod_{i=2}^{m-1} \rho(w_1+w_2)^{\sigma^i}\right) \rho(w_2)^{\sigma^{m+1}}\\
&\quad + 2 \prod_{i=1}^{m} \rho(w_1+w_2)^{\sigma^i}
\end{aligned} \tag{7.17}$$

is the sum of terms in (7.15) with length $2m$, when restricted to $G = G(m)$ (or equivalently, to $G(n)$). We note that the last three terms in (7.17) correspond to cases 4–6 of (7.13). Actually, to get case 4 of (7.13), we have to let $\mu(k) = \rho(w_1)\,\rho(w_2)^\sigma$, $\xi(k) = \rho(w_2)\,\rho(w_1)^\sigma$, $\lambda(k) = \rho(w_1 + w_2)\,\rho(w_1)^\sigma$ for $0 \leqslant k \leqslant m-1$. Similarly, to get case 5 of (7.13), we have to let $\mu(k) = \rho(w_2)\,\rho(w_1)^\sigma$, $\xi(k) = \rho(w_1)\,\rho(w_2)^\sigma$, $\lambda(k) = \rho(w_1 + w_2)\,\rho(w_1)^\sigma$ for $0 \leqslant k \leqslant m-1$. To get case 6 of (7.13), we have to let either (i) $\mu(k) = \xi(k) = \rho(w_2)\,\rho(w_1)^\sigma$ or (ii) $\mu(k) = \xi(k) = \rho(w_1)\,\rho(w_2)^\sigma$ and $\lambda(k) = \rho(w_1 + w_2)^\sigma$ for $0 \leqslant k \leqslant m-1$. This is the reason why the coefficient of the last term in (7.17) is 2.

If we define, for $1 \leqslant i, j \leqslant 12$ and $1 \leqslant t \leqslant m$,

$$q(\mathfrak{R}(i), \mathfrak{E}(j), t) = \sum a(\mu, \xi, \lambda, I), \tag{7.18}$$

where the summation is the same as in (7.16), then (7.1) = (7.14) is equal to

$$\sum_{i=1}^{12} q(\mathfrak{R}(i), \mathfrak{E}(i), m) + 4. \tag{7.19}$$

Our next step is to express (7.18) in an explicit form. As in Section 6, we will find some linear recursive equations satisfied by the left-hand side of (7.18).

LEMMA 7.20. *For a fixed* $1 \leqslant 12$, *let* q_t *be the column vector*

$$(q(\mathfrak{R}(i), \mathfrak{E}(j), t))_{j=1}^{12}$$

with $t = 1, 2, \ldots, m$. *Then* $q_{t+1} = X_2 q_t$, *where*

$$X_2 = \begin{pmatrix} U_2 & V_2 & 0 & 0 \\ V_2 & U_2 & 0 & 0 \\ 0 & 0 & W_2 & Y_2 \\ 0 & 0 & Y_2 & W_2 \end{pmatrix},$$

and

$$U_2 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 6 & 0 & 0 & 6 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 4 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 2 & 0 \\ 6 & 3 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & 0 \\ 6 & 4 \end{pmatrix}.$$

Moreover, $q_m = X_2^{m-1} q_1$.

Proof. The method we will use to construct X_2 is similar to that in Lemma 6.19. Suppose we want to express $Q(\mathfrak{R}(i), \mathfrak{E}(4), t+1)$ in terms of $Q(\mathfrak{R}(i), \mathfrak{E}(j), t)$ with $j = 1, 2, \ldots, 12$; $\mathfrak{E}(4) = (\rho(w_1), \rho(w_2), \rho(w_2))$. For every μ (resp. ξ) $\in \mathfrak{L}(A_{t+1})$, there exists exactly one μ' (resp. ξ') $\in \mathfrak{L}(A_t)$ such that $\mu(k) = \mu'(k)$ (resp. $\xi'(k) = \xi(k)$) for $0 \leqslant k \leqslant t-1$. $\mu(t)$ (resp. $\xi(t)$) should be chosen in the following way:

(i) For $Q(\mathfrak{R}(i), (\rho(w_1), \rho(w_1), *), t)$, we choose

$$\mu(t) = \rho(w_2)\,\rho(w_1)^\sigma,\ \xi(t) = 2\rho(w_2)\,\rho(w_2)^\sigma.$$

(ii) For $Q(\mathfrak{R}(i), (\rho(w_1), \rho(w_2), *), t)$, we choose

$$\mu(t) = \rho(w_2)\,\rho(w_1)^\sigma,\ \xi(t) = \rho(w_1)\,\rho(w_2)^\sigma.$$

(iii) For $Q(\mathfrak{R}(i), (\rho(w_2), \rho(w_1), *), t)$, we choose

$$\mu(t) = 2\rho(w_1)\,\rho(w_1)^\sigma,\ \xi(t) = 2\rho(w_2)\,\rho(w_2)^\sigma.$$

(iv) For $Q(\mathfrak{R}(i), (\rho(w_2), \rho(w_2), *), t)$, we choose

$$\mu(t) = 2\rho(w_1)\,\rho(w_1)^\sigma,\ \xi(t) = \rho(w_1)\,\rho(w_2)^\sigma.$$

Once we get $\mu(k)$ and $\xi(t)$, $I(\mu)(t)$ and $I(\xi)(t)$ are also determined by definition. For every $\lambda \in \mathfrak{M}(\mu, \xi)$, there exists exactly one $\lambda' \in \mathfrak{M}(\mu', \xi')$ such that $\lambda(k) = \lambda'(k)$ for all $k = 0, 1, \ldots, t-1$; $\lambda(t)$ is restricted by (7.6). More precisely, $\lambda(t)$ should be chosen in the following way: For $Q(\mathfrak{R}(i), (\rho(w_1), \rho(w_1), *), t)$, $I(\mu)(t) = (1, 0)$, $I(\xi)(t) = (1, 1)$, we choose $\lambda(t) \in E_5$, i.e., $\lambda(t) = 2\rho(w_1)\,\rho(w_2)^\sigma$, $4\rho(w_1)^\sigma$ or $6\rho(w_2)$. For $Q(\mathfrak{R}(i), (\rho(w_1), \rho(w_1), 1), t)$ or $Q(\mathfrak{R}(i), (\rho(w_1), \rho(w_1), \rho(w_1)), t)$, it is impossible to define $\lambda(t)$ as above so that

$$\prod_{i=0}^{t-1} \lambda(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(\mu, \xi, \lambda, I)\,\rho_I$$

has a nontrivial term satisfying condition (iii) of (7.16). Now, for $Q(\mathfrak{R}(i), (\rho(w_1), \rho(w_1), \rho(w_2)), t)$, we have to choose $\lambda(t) = 2\rho(w_1)\,\rho(w_2)^\sigma$. Similar consideration gives

$$\begin{aligned} &Q(\mathfrak{R}(i), \mathfrak{E}(4), t+1) \\ &\quad = 2Q(\mathfrak{R}(\mathrm{i}), \mathfrak{E}(2), t) + Q(\mathfrak{R}(i), \mathfrak{E}(3), t) + 2Q(\mathfrak{R}(i), \mathfrak{E}(5), t). \end{aligned}$$

This describes one method to construct X_2. A similar way as in the second part of the proof of Lemma 6.19 is more effective. We leave it to the reader.

LEMMA 7.21. *Let X_3 be the matrix whose* jth *column is the column vector*

$$(q(\mathfrak{R}(j), \mathfrak{E}(i), t))_{i=1}^{12}$$

Then $X_3 = X_2$.

Proof. This is an easy computation. We leave it to the reader.

By (7.19), (7.1) is equal to $\mathrm{tr}(X_2^{m-1}X_3) + 4 = \mathrm{tr}\, X_2^m + 4$. The eigenvalues of X_2 are 2, 2, e, f, -2, -2, $-e$, $-f$, -1, 2, 2, 7, where e, f are two roots of $x^2 - 8x + 10 = 0$. As a result, we get

PROPOSITION 7.22.

$$\sum_{T' \neq I, J \in \Omega} (\Gamma\eta_T, \rho_I)(\Gamma\eta_T, \rho_J)\langle \Gamma\eta_{I'}, \Gamma\eta_{J'}\rangle$$
$$= 4 + (-1)^m + 2^{m+1} + 7^m + (2(2^m) + e^m + f^m)$$
$$+ [2(-2)^m + (-e)^m + (-f)^m],$$

where e, f are two roots of $x^2 - 8x + 10 = 0$.

8. COMPUTATION OF $C_{\phi,\varphi}$ CONCLUDED

In this section, we will compute

$$\sum_{T' \neq J \in \Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle + \sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle. \tag{8.1}$$

We first show that the two summations in (8.1) are equal.

LEMMA 8.2. *For $I \in \Omega$, $(\Gamma\eta_T, \rho_I) = (\Gamma\eta_T, \rho_{\bar{I}})$.*

Proof. Since $(\Gamma\eta_T, \rho_I)$ is an integer and Γ, η_T are real functions, so $(\Gamma\eta_T, \rho_I) = (\Gamma_{\eta_T}, \rho_{\bar{I}})$;

$$\sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle = \sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_{\bar{I}})\langle \Gamma\eta_{\bar{I}'}, \Gamma\eta_T\rangle$$
$$= \sum_{T' \neq J \in \Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle. \tag{8.3}$$

This shows the two summations in (8.1) are equal. In the following we will compute

$$\sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma, \eta_{I'}\, \eta_T\Gamma\rangle. \tag{8.4}$$

By (7.2), (8.4) is equal to

$$\sum_{T'\neq I\in\Omega}\ \sum_{\substack{\mu\in\mathfrak{L}\\ I(\mu)=I}} \langle\Gamma,[\mu]\rangle\langle\Gamma,\eta_{I(\mu)'}\eta\Gamma\rangle$$

$$= \sum_{\mu\in\mathfrak{L}} \langle\Gamma,[\mu]\rangle\langle\Gamma,\eta_{I(\mu)'}\eta_T\Gamma\rangle = \sum_{\mu\in\mathfrak{L}} 2^{t(\mu)}\langle\Gamma,\eta_{I(\mu)'}\eta_T\Gamma\rangle$$

$$= \left\langle\Gamma, \sum_{\mu\in\mathfrak{L}} (2^{t(\mu)}\eta_{I(\mu)'})\,\eta_T\Gamma\right\rangle. \tag{8.5}$$

By definition,

$$2^{t(\mu)}\eta_{I(\mu)'}\,\eta_T = 2^{t(\mu)}\prod_{i=0}^{m-1} (\eta(I(\mu)'_1(i)\,w_1 + I(\mu)'_2(i)\,w_2)$$
$$\times\,\eta(w_1+w_2)\,\rho(w_1+w_2))^{\sigma^i}.$$

By a routine computation using Lemma 3.2, we get

$$2\eta(0)\,\eta(w_1+w_2)\,\rho(w_1+w_2)$$
$$= 2\rho(w_1+w_2)^\sigma + 4\rho(w_1)\,\rho(w_1)^\sigma + 4\rho(w_2)\,\rho(w_2)^\sigma + 8, \tag{8.6.1}$$
$$\eta(w_1)\,\eta(w_1+w_2)\,\rho(w_1+w_2)$$
$$= \rho(w_1)\,\rho(w_1+w_2)^\sigma + 2\rho(w_1+w_2)\,\rho(w_2)^\sigma$$
$$+\,4\rho(w_2)\,\rho(w_1)^\sigma + 2\rho(w_1)^{\sigma^2} + 6\rho(w_2)^\sigma + 4\rho(w_1), \tag{8.6.2}$$
$$\eta(w_2)\,\eta(w_1+w_2)\,\rho(w_1+w_2)$$
$$= \rho(w_2)\,\rho(w_1+w_2)^\sigma + 2\rho(w_1+w_2)\,\rho(w_1)^\sigma$$
$$+\,4\rho(w_1)\,\rho(w_2)^\sigma + 2\rho(w_2)^{\sigma^2} + 6\rho(w_1)^\sigma + 4\rho(w_2). \tag{8.6.3}$$

For $A\subset\{0,1,2,..., m-1\}\subset S(n)$ and $\mu\in\mathfrak{L}(A)$, let $\mathfrak{N}(\mu)$ be the set of all functions $\xi\colon A\to F=F_1\cup F_2\cup F_3$, where F_i is the set of terms in (8.6.i) for $1\leqslant i\leqslant 3$, (e.g., $F_2=\{\rho(w_1)\,\rho(w_1+w_2)^\sigma,\ 2\rho(w_1+w_2)\,\rho(w_2)^\sigma,\ 4\rho(w_2)\ \rho(w_1)^\sigma,\ 2\rho(w_1)^{\sigma^2},\ 6\rho(w_2)^\sigma,\ 4\rho(w_1)\}$), and for all $k\in A$, $\xi(k)$ satisfies the conditions

$$\begin{aligned}
\xi(k)&\in F_1, &&\text{if}\quad I(\mu)(k)=(1,1), &&\text{i.e.,}\quad \mu(k)=2\rho(w_1)\,\rho(w_1)^\sigma\\
&&&&&\qquad\text{or } 2\rho(w_2)\,\rho(w_2)^\sigma, \qquad (8.7)\\
&\in F_2, &&\text{if}\quad I(\mu)(k)=(1,0), &&\text{i.e.,}\quad \mu(k)=\rho(w_1)\,\rho(w_2)^\sigma,\\
&\in F_3, &&\text{if}\quad I(\mu)(k)=(0,1), &&\text{i.e.,}\quad \mu(k)=\rho(w_2)\,\rho(w_1)^\sigma.
\end{aligned}$$

For $\mu\in\mathfrak{L}=\mathfrak{L}(S)$, we can get easily from this definition that

$$2^{t(\mu)}\eta_{I(\mu)'}\eta_T\Gamma = \sum_{\xi\in\mathfrak{N}(\mu)}\prod_{i=0}^{m-1}\xi(i)^{\sigma^i}|_G. \tag{8.8}$$

Let

$$\mathfrak{X} = \left\{\begin{vmatrix}0&0\\0&0\end{vmatrix}, \begin{vmatrix}1&0\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\1&0\end{vmatrix}, \begin{vmatrix}1&0\\1&0\end{vmatrix}, \begin{vmatrix}0&1\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\0&1\end{vmatrix}, \begin{vmatrix}0&1\\1&0\end{vmatrix}, \begin{vmatrix}1&0\\0&1\end{vmatrix}\right\}.$$

The proofs of Lemmas 8.9–8.11 are left to the reader.

LEMMA 8.9. *Let $\xi_0, \xi_1 \in F$. Then $\xi_0 \xi_1^{\sigma}$ is of 2 type*

$$\mathfrak{X} \cup \left\{\begin{vmatrix}1&1\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\1&1\end{vmatrix}\right\}.$$

LEMMA 8.10. *Suppose $1 \leqslant k \leqslant m$ and $\xi(0), \xi(1),..., \xi(k-1) \in F$. Then*

$$\prod_{i=0}^{k-1} \xi(i)^{\sigma^i}$$

is of k type

$$\mathfrak{X} \cup \left\{\begin{vmatrix}1&1\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\1&1\end{vmatrix}, \begin{vmatrix}0&1\\0&1\end{vmatrix}, \begin{vmatrix}1&1\\1&0\end{vmatrix}, \begin{vmatrix}1&0\\1&1\end{vmatrix}\right\}.$$

LEMMA 8.11. *Let*

$$\xi \in \mathfrak{X} \cup \left\{\begin{vmatrix}1&1\\0&0\end{vmatrix}, \begin{vmatrix}0&0\\1&1\end{vmatrix}\right\}.$$

Suppose $3 \leqslant k \leqslant m$, $\xi(2), \xi(3),..., \xi(k-1) \in F$ and

$$\xi^{\sigma^2} \prod_{i=2}^{k-1} \xi(i)^{\sigma^i} = \sum_{I \in \Omega(n)} b(I)\, \rho_I,$$

with $b(I) \in Z$. If $b(I) \neq 0$ and ρ_I is $(2, k-1)$ complete, then ρ_I is of k type $\mathfrak{X}$.

LEMMA 8.12. *Let $3 \leqslant k \leqslant m$, $\xi(0), \xi(1),..., \xi(k-1) \in F$ and*

$$\prod_{i=0}^{k-1} \xi(i)^{\sigma^i} = \sum_{I \in \Omega(n)} b(I)\, \rho_I,$$

with $b(I) \in Z$. If $b(I) \neq 0$ and ρ_I is $(2, k-1)$ complete, then ρ_I is of k type $\mathfrak{X}$.

Proof. This comes from Lemmas 8.9 and 8.11.

Let $\xi(0), \xi(1),..., \xi(m-1) \in F$ and

$$\prod_{i=0}^{m-1} \xi(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(I)\, \rho_I,$$

with $a(I) \in Z$. Suppose $a(I) \neq 0$ and $\rho_I|_G$ has length $2m$. Then ρ_I is $(2, m-1)$ complete by Lemma 8.10. Lemma 8.12 shows that ρ_I is of m type $\mathfrak{X}$. This implies that the 2 pattern of ρ_I is

$$\mathfrak{Y} = \left\{ \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right\}.$$

For $2 \leqslant k \leqslant m$, let

$$\prod_{i=0}^{k-1} \xi(i)^{\sigma^i} = \sum_{J \in \Omega(n)} b(J, k)\, \rho_J,$$

with $b(J, k) \in Z$. There exists $J^k \in \Omega(n)$ with $b(J^k, k) \neq 0$ so that $x\rho_I$ is a nontrivial term of

$$\rho_{J^k} \prod_{i=k}^{m-1} \xi(i)^{\sigma^i},$$

where x is a constant. Of course, such a J^k is not unique. Since ρ_I is $(2, m-1)$ complete, so ρ_{J^k} is $(2, k-1)$ complete. Also, ρ_{J^k} is of 2 pattern $\mathfrak{Y}$. Once again, by some routine computations and induction on k, we can get Lemma 8.13. The detail is left to the reader.

LEMMA 8.13. *The notation is as above. Then*

(i) *if* ρ_{J^k} *is of* k *type* $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$, *then* $\xi(i-1) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ *and* ρ_{J^i} *is of* i *type* $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ *for all* $1 \leqslant i \leqslant k$.

(ii) *if* ρ_{J^k} *is of* k *type* $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, *then* $\xi(i-1) = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$ *and* ρ_{J^i} *is of* i *type* $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ *for all* $1 \leqslant i \leqslant k$.

In summary, we get that ρ_I is $(2, m-1)$ complete and one of the following 8 cases holds:

	2 pattern of ρ_I	m type of ρ_I		2 pattern of ρ_I	m type of ρ_I
1	$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$	5	$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$
2	$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$	6	$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$
3	$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$	7	$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$
4	$\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$ or $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$	8	$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

(8.14)

The last two cases are discussed in Lemma 8.13.

Now, by (8.5) and (8.8) we get that (8.4) is equal to

$$\left\langle \Gamma, \sum_{\mu \in \mathfrak{L}} \sum_{\xi \in \mathfrak{N}(\mu)} \prod_{i=0}^{m-1} \xi(i)^{\sigma^i} \right\rangle. \tag{8.15}$$

To compute (8.15), we have to study those terms in

$$\sum_{\mu \in \mathfrak{L}} \sum_{\xi \in \mathfrak{N}(\mu)} \prod_{i=0}^{m-1} \xi(i)^{\sigma^i} \tag{8.16}$$

whose restriction to G have length $2m$. For $1 \leqslant t \leqslant m$, let $A_t = \{0, 1, 2, ..., t-1\} \subset S(n)$. For $\mu \in \mathfrak{L}(A_t)$ and $\xi \in \mathfrak{N}(\mu)$, let

$$\prod_{i=0}^{t-1} \xi(i)^{\sigma^i} = \sum_{I \in \Omega(n)} a(\mu, \xi, I)\, \rho_I$$

with $a(\mu, \xi, I) \in Z$. Define (cf. Lemma 8.21)

$$\begin{aligned}
\mathfrak{Y}(1) &= \left(\rho(w_2), \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right), & \mathfrak{X}(1) &= \left(\rho(w_1), \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(2) &= \left(\rho(w_2), \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right), & \mathfrak{X}(2) &= \left(\rho(w_1), \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(3) &= \left(\rho(w_2), \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\right), & \mathfrak{X}(3) &= \left(\rho(w_1), \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(4) &= \left(\rho(w_2), \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right), & \mathfrak{X}(4) &= \left(\rho(w_1), \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(5) &= \left(\rho(w_1), \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right), & \mathfrak{X}(5) &= \left(\rho(w_2), \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(6) &= \left(\rho(w_1), \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\right), & \mathfrak{X}(6) &= \left(\rho(w_2), \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(7) &= \left(\rho(w_1), \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}\right), & \mathfrak{X}(7) &= \left(\rho(w_2), \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(8) &= \left(\rho(w_1), \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right), & \mathfrak{X}(8) &= \left(\rho(w_2), \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\right),\\
\mathfrak{Y}(9) &= \left(\rho(w_2), \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\right), & \mathfrak{X}(9) &= \left(\rho(w_1), \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(10) &= \left(\rho(w_2), \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right), & \mathfrak{X}(10) &= \left(\rho(w_1), \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}\right),\\
\mathfrak{Y}(11) &= \left(\rho(w_1), \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right), & \mathfrak{X}(11) &= \left(\rho(w_2), \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\right),\\
\mathfrak{Y}(12) &= \left(\rho(w_1), \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\right), & \mathfrak{X}(12) &= \left(\rho(w_2), \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\right).
\end{aligned}$$

We also write $\mathfrak{Y}(i) = (\mathfrak{Y}(i)_1, \mathfrak{Y}(i)_2)$ and $\mathfrak{X}(i) = (\mathfrak{X}(i)_1, \mathfrak{X}(i)_2)$ for $1 \leqslant i \leqslant 12$. For fixed $1 \leqslant i, j \leqslant 12$ and $1 \leqslant t \leqslant m$, let

$$R(\mathfrak{Y}(i), \mathfrak{X}(j), t) = \sum a(\mu, \xi, I)\, \rho_I, \tag{8.17}$$

where the summation ranges over all $\mu \in \mathfrak{L}(A_t)$, $\xi \in \mathfrak{N}(\mu)$ such that

(i) the only term of length $2t$ (considered as in $G(n)$) in

$$[\mu] = \prod_{i=0}^{t-1} \mu(i)^{\sigma^i}$$

is of 2 pattern $\mathfrak{Y}(i)_1$ and is of t type $\mathfrak{X}(i)_1$,

(ii) ρ_I is of 2 pattern $\mathfrak{Y}(i)_2$, of t type $\mathfrak{X}(i)_2$ and is $(2, t-1)$ complete.

By the above discussion, we get that

$$\begin{aligned} &\sum_{i=1}^{12} R(\mathfrak{Y}(i), \mathfrak{X}(i), m) + R(\mathfrak{Y}(1), \mathfrak{X}(3), m) + R(\mathfrak{Y}(5), \mathfrak{X}(7), m) \\ &\quad + \rho(w_1)\, \rho(w_2)^{\sigma} \left(\prod_{i=2}^{m-1} \rho(w_1 + w_2)^{\sigma^i} \right) \rho(w_2)^{\sigma^m} \rho(w_1)^{\sigma^{m+1}} \\ &\quad + \rho(w_2)\, \rho(w_1)^{\sigma} \left(\prod_{i=2}^{m-1} \rho(w_1 + w_2)^{\sigma^i} \right) \rho(w_1)^{\sigma^m} \rho(w_2)^{\sigma^{m+1}} \end{aligned} \tag{8.18}$$

is the sum of terms in (8.16) with length $2m$, when restricted to $G = G(m)$ (or equivalently, to $G(n)$). We note that the last two terms in (8.18) correspond to cases 7, 8 of (8.14). Actually, to get case 7 of (8.14), we have to let $\mu(k) = \rho(w_2)\, \rho(w_1)^{\sigma}$ and $\xi(k) = \rho(w_1)\, \rho(w_1 + w_2)^{\sigma}$ for all $k = 0, 1, \ldots, m-1$. Similarly, to get case 8 of (8.14), we have to let $\mu(k) = \rho(w_1)\, \rho(w_2)^{\sigma}$ and $\xi(k) = \rho(w_2)\, \rho(w_1 + w_2)^{\sigma}$ for all $k = 0, 1, \ldots, m-1$.

If we define, for $1 \leqslant i, j \leqslant 12$ and $2 \leqslant t \leqslant m$,

$$r(\mathfrak{Y}(i), \mathfrak{X}(j), t) = \sum a(\mu, \xi, I), \tag{8.19}$$

where the summation is the same as in (8.17), then (8.4) = (8.15) is equal to

$$\begin{aligned} &\sum_{i=1}^{12} r(\mathfrak{Y}(i), \mathfrak{X}(i), m) + 2r(\mathfrak{Y}(1), \mathfrak{X}(3), m) \\ &\quad + 2r(\mathfrak{Y}(5), \mathfrak{X}(7), m) + 2. \end{aligned} \tag{8.20}$$

Our next step is to express (8.20) in an explicit form. As in Sections 6 and 7, we will find some linear recursive equations satisfied by the left-hand side of (8.20).

LEMMA 8.21. *For a fixed* $1 \leqslant i \leqslant 12$, *let* r_t *be the column vector*

$$(r(\mathfrak{Y}(i), \mathfrak{X}(j), t))_{j=1}^{12}$$

with $t = 1, 2, \ldots, m$. *Then* $r_{t+1} = X_4 r_t$, *where*

$$X_4 = \begin{pmatrix} U_4 & V_4 & 0 & 0 \\ V_4 & U_4 & 0 & 0 \\ 0 & 0 & W_4 & Y_4 \\ 0 & 0 & Y_4 & W_4 \end{pmatrix},$$

and

$$U_4 = \begin{pmatrix} 0 & 4 & 0 & 2 \\ 2 & 0 & 10 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \qquad V_4 = \begin{pmatrix} 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_4 = \begin{pmatrix} 4 & 4 \\ 0 & 2 \end{pmatrix}, \qquad Y_4 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, $r_m = X_4^{m-2} r_2$.

Proof. The proof is the same as those in Lemmas 6.19 and 7.20. We leave it to the reader.

LEMMA 8.22. *Let* X_5 *be the matrix whose* jth *column is the column vector*

$$(r(\mathfrak{Y}(j), \mathfrak{X}(i), 2))_{i=1}^{12}.$$

Then

$$X_5 = \begin{pmatrix} U_5 & V_5 & 0 & 0 \\ V_5 & U_5 & 0 & 0 \\ 0 & 0 & W_5 & Y_5 \\ 0 & 0 & Y_5 & W_5 \end{pmatrix},$$

and

$$U_5 = \begin{pmatrix} 8 & 0 & 44 & 0 \\ 0 & 34 & 0 & 24 \\ 2 & 0 & 14 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}, \qquad V_5 = \begin{pmatrix} 0 & 24 & 0 & 16 \\ 8 & 0 & 60 & 0 \\ 0 & 6 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$

$$W_5 = \begin{pmatrix} 32 & 24 \\ 0 & 4 \end{pmatrix}, \qquad Y_5 = \begin{pmatrix} 32 & 16 \\ 0 & 0 \end{pmatrix}.$$

Proof (cf. Lemmas 6.20 and 7.21). We leave the proof to the reader.

Let X_6 be the matrix whose entries are the same as X_5 except that (i) the third column vector of X_6 is the sum of the third one of X_5 and twice the first one of X_5, (ii) the seventh column vector of X_6 is the sum of the seventh one of X_5 and twice the fifth one of X_5. That is

$$X_6 = \begin{pmatrix} U_6 & V_6 & 0 & 0 \\ V_6 & U_6 & 0 & 0 \\ 0 & 0 & W_5 & Y_5 \\ 0 & 0 & Y_5 & W_5 \end{pmatrix},$$

$$U_6 = \begin{pmatrix} 8 & 0 & 60 & 0 \\ 0 & 34 & 0 & 24 \\ 2 & 0 & 18 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}, \qquad V_6 = \begin{pmatrix} 0 & 24 & 0 & 16 \\ 8 & 0 & 76 & 0 \\ 0 & 6 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix}.$$

Now, (8.20) is equal to $\operatorname{tr}(X_4^{m-2}X_6) + 2$.

A mysterious phenomenon for which we cannot give a conceptual explanation is $X_6 = X_4^2$. This is similar to the situation in Section 6. An intrinsic proof of this will be helpful to avoid the tedious computation in Lemma 6.20. By the fact $X_6 = X_4^2$, we get

$$\operatorname{tr}(X_4^{m-2}X_6) = \operatorname{tr} X_4^m,$$

which is the sum of the mth powers of the eigenvalues of X_4. The eigenvalues of X_4 are -2, α, β, γ, 2, $-\alpha$, $-\beta$, $-\gamma$, 8, 2, 0, 2, where α, β, and γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$. As a result, we get

PROPOSITION 8.23.

$$\sum_{T' \neq J \in \Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle + \sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle$$
$$= 4 + 2^{3m+1} + 2^{m+2} + 2(2^m + \alpha^m + \beta^m + \gamma^m)$$
$$+ 2[(-2)^m + (-\alpha)^m + (-\beta)^m + (-\gamma)^m],$$

where α, β, γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$.

THEOREM 8.24.

$$C_{\phi.\phi} = \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)}\rangle$$
$$= 7^m + (-1)^m - 2^{3m+1} + a^m + b^m$$

$$+ [2^m + c^m + d^m] + [(-2)^m + (-c)^m + (-d)^m]$$
$$+ [e^m + f^m] + [(-e)^m + (-f)^m]$$
$$- 2[\alpha^m + \beta^m + \gamma^m] - 2[(-\alpha)^m + (-\beta)^m + (-\gamma)^m],$$

where a, b, are two roots of $x^2 - 12x + 24 = 0$, c, d are two roots of $x^2 - 8x - 8 = 0$, e, f are two roots of $x^2 - 8x + 10 = 0$, and α, β, γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$.

9. The Projective Indecomposable Characters of $SU_3(2^m)$

In the rest of this paper, we consider the principal indecomposable characters and the first Cartan invariant in characteristic 2 of $G = SU_3(2^m)$. We will use the same notation as in Section 2 except that $G(n) = SU_3(2^n)$ and $S(n) = \{0, 1, 2, \dots, n-1\} \subset Z$. This also means that $G = SU_3(2^m)$ and $S = \{0, 1, 2, \dots, m-1\} \subset Z$. Since the method is the same as that used in previous sections, we will only sketch the proof and point out those different steps.

Lemma 9.1. $\rho(w_1)^{\sigma m} = \rho(w_2)$ and $\rho(w_2)^{\sigma m} = \rho(w_1)$.

Proof. See [2].

It is easy to see that all results except Theorem 3.4 in Section 3 are true for $G = SU_3(2^m)$.

Theorem 9.2. *The Grothendieck ring $R(G)$ of G in characteristic 2 is isomorphic to the commutative Z algebra which is generated by elements x_i, y_i, z_i, $i \in S$ that satisfy*

$$\begin{aligned}
x_i^2 &= x_{i+1} + 2y_i, && 0 \leqslant i < m-1,\\
y_i^2 &= y_{i+1} + 2x_i, && 0 \leqslant i < m-1,\\
x_i y_i &= z_i + 1, && 0 \leqslant i \leqslant m-1,\\
x_{m-1}^2 &= y_0 + 2y_{m-1}, && .\\
y_{m-1}^2 &= x_0 + 2x_{m-1}.
\end{aligned}$$

Proof (cf. Theorem 3.4).

To interpet the results in Section 4 for $G = SU_3(2^m)$, we make the convention

$$\begin{aligned}
\mu(j) &= \overline{\mu(j-m)}, && \text{if} \quad m \leqslant j < 2m, \\
I(j) &= \overline{I(j-m)}, && \text{if} \quad m \leqslant j < 2m, \\
\mu(j) &= \overline{\mu(j+m)}, && \text{if} \quad -m \leqslant j < 0, \\
I(j) &= \overline{I(j+m)}, && \text{if} \quad -m \leqslant j < 0.
\end{aligned} \tag{9.3}$$

By these and Lemma 9.1, we can get that all results before Corollary 4.20 in Section 4 are true for $G = SU_3(2^m)$.

COROLLARY 9.4. *Suppose $I, J \in \Omega$, $I \neq J'$, and $\langle \Gamma\eta_I, \rho_J \rangle \neq 0$. Then there exists $k \in S$ such that $I(k) = J(k) = (1, 1)$ and hence I and J are well behaved.*

THEOREM 9.5. *Suppose $I \in \Omega$. Then*

$$\Phi_{I'} = \Gamma\eta_I - \sum_{I' \neq J \in \Omega} \langle \Gamma\eta_I, \rho_J \rangle \, \Gamma\eta_{J'}.$$

Proof. This follows from Corollaries 4.17, 4.18, and 9.3.

To study the degrees of the principal indecomposable characters, we follow the discussion in Section 5. We note that for $I, J \in \Omega$, we define

$$\begin{aligned}
(\Gamma\eta_I, \rho_J) &= \langle \Gamma\eta_T, \rho_T \rangle - 1, && \text{if} \quad I = J = T, \\
&= \langle \Gamma\eta_I, \rho_J \rangle, && \text{otherwise.}
\end{aligned}$$

The reason for defining this notation has been explained in Section 5. By this notation and Theorem 9.5, we have

$$\Phi_{(\phi,\phi)} = \Gamma\eta_T - \Gamma - \sum_{(\phi,\phi) \neq I \in \Omega} (\Gamma\eta_T, \rho_I) \, \Gamma\eta_{I'}. \tag{9.6}$$

LEMMA 9.7. (i) *Suppose $I \in \Omega$ and $I \neq T$. Then*

$$\Phi_{I'}(1_G) = \Gamma(1_G)[\eta_I(1_G) - \sum_{I' \neq J \in \Omega} (\Gamma\eta_I, \rho_J) \, \eta_{J'}(1_G)].$$

(ii) $\Phi_{(\phi,\phi)}(1_G) = \Gamma(1_G)[\eta_T(1_G) - 1 - \sum_{I' \neq J \in \Omega} (\Gamma\eta_I, \rho_J) \, \eta_{J'}(1_G)]$.

To compute $\Phi_{I'}(1_G)$, we need only consider the term

$$\sum_{I' \neq J \in \Omega} (\Gamma\eta_I, \rho_J) \, \eta_{J'}(1_G). \tag{9.8}$$

Following the discussion in Section 5, we can get easily that (9.8) is equal to $Q'_m + P_m = 5^m - 1$ for $I = T$. Hence, $\Phi_{(\phi,\phi)}(1_G) = 2^{3m}(6^m - 5^m)$ by Lemma 9.7(ii). Now suppose that I' is non well behaved and $I \neq T$. Then I satisfies the following three conditions:

(i) $I(k) \neq (0,0)$ for all $k \in S$,

(ii) if $I(k) = (1,0)$ for some $k \in S$, then $I(k+1) = (1,0)$ or $(1,1)$.

(iii) If $I(k) = (0,1)$ for some $k \in S$, then $I(k+1) = (0,1)$ or $(1,1)$.

Let $I_1 \cap I_2 = \bigcup_{s=1}^{e} B_s$, where

(i) $B_s = \{i_s, i_s+1, i_s+2, ..., j_s\}$ for some $i_s, j_s \in S$,

(ii) $i_s - 1, j_s + 1 \notin I_1 \cap I_2$,

(iii) $0 \leqslant i_1 < i_2 < \cdots < i_e \leqslant m-1$.

If $i_1 \neq 0$ or $j_e \neq m-1$, we define

$$\begin{aligned} \varepsilon_s &= 0, \qquad \text{if} \quad I(i_s - 1) = I(j_s + 1), \\ &= 1, \qquad \text{if} \quad I(i_s - 1) \neq I(j_s + 1), \end{aligned}$$

and $|B_s| = b_s$.

If $i_1 = 0$ and $j_e = m-1$, we define ε_s and b_s as above for $2 \leqslant s \leqslant e$ and define

$$\begin{aligned} \varepsilon_1 &= 0, \qquad \text{if} \quad I(i_e - 1) = I(j_1 + 1), \\ &= 1, \qquad \text{if} \quad I(i_e - 1) \neq I(j_1 + 1), \end{aligned}$$

and $b_1 = |B_1| + |B_e|$, $\varepsilon_e = b_e = 0$; I is said to be of type $\{(b_s, \varepsilon_s) | 1 \leqslant s \leqslant e\}$.

By the above notation and the discussion in Section 5, we get

THEOREM 9.9 ([2, Theorem A, (7.6), Corollary, Theorem 8.2]). (i) $\Phi_{(\phi,\phi)}(1_G) = 2^{3m}(6^m - 5^m)$.

(ii) *If $I = (I_1, I_2) \in \Omega$ is well behaved, then*

$$\Phi_I(1_G) = 2^{3m + |I_1' \cap I_2'|}\, 3^{|I_1' \cup I_2'|}.$$

(iii) *If $I = (I_1, I_2) \in \Omega$ is non well behaved and $I' \neq I$ is of type $\{(b_s, \varepsilon_s) | 1 \leqslant s \leqslant e\}$, then*

$$\Phi_I(1_G) = 2^{3m} \left\{ 2^{|I_1' \cap I_2'|}\, 3^{|I_1' \cup I_2'|} - \prod_{s=1}^{e} \left(\frac{5^{b_s} + (-1)^{\varepsilon_s}}{2} \right) \right\}.$$

10. THE FIRST CARTAN INVARIANT OF $SU_3(2^m)$

In this section, we will compute $C_{\phi,\phi} = \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)} \rangle$. By (9.6) and the fact $\langle \Gamma, \Phi_{(\phi,\phi)} \rangle = 0$, we have

$$\begin{aligned} C_{\phi,\phi} &= \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)} \rangle \\ &= \langle \Gamma\eta_T, \Phi_{(\phi,\phi)} \rangle - \sum_{(\phi,\phi) \neq I \in \Omega} (\Gamma\eta_T, \rho_I) \langle \Gamma\eta_{I'}, \Phi_{(\phi,\phi)} \rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle \Gamma\eta_T, \Gamma\eta_T\rangle - \langle \Gamma\eta_T, \Gamma\rangle + \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\rangle \\
&\quad - \sum_{(\phi,\phi)\neq J\in\Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle \\
&\quad - \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle \\
&\quad + \sum_{(\phi,\phi)\neq I,J\in\Omega} (\Gamma\eta_T, \rho_I)(\Gamma\eta_T, \rho_J)\langle \Gamma\eta_{I'}, \Gamma\eta_{J'}\rangle. \qquad (10.1)
\end{aligned}
$$

By Corollary 9.4, we get (cf. Lemma 6.2)

LEMMA 10.2. *Let $I\in\Omega$ and $I\neq(\phi,\phi)$. Then*

$$
\begin{aligned}
(\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\rangle &= \langle \Gamma\eta_T, \Gamma\rangle - 1, && \textit{if} \quad I = T, \\
&= 0, && \textit{otherwise.}
\end{aligned}
$$

This lemma gives

$$-\langle \Gamma\eta_T, \Gamma\rangle + \sum_{(\phi,\phi)\neq I\in\Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\rangle = -1. \qquad (10.3)$$

To compute $\langle \Gamma\eta_T, \Gamma\eta_T\rangle = \langle \Gamma, \eta_T^2\Gamma\rangle$ in (10.1), we note that Lemmas 6.9–6.13 in Section 6 are true for $G = SU_3(2^m)$. The proof of these lemmas for $G = SU_3(2^m)$ is exactly the same as that in Section 6 although there should be a slight change in the proof of Lemma 6.12.

For $G = SU_3(2^m)$, tabulation (6.15) in Lemma 6.14 should be changed to

	2 pattern of ρ_I	m type of ρ_I		2 pattern of ρ_I	m type of ρ_I
1	$\begin{vmatrix}1&1\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\0&0\end{vmatrix}$	7	$\begin{vmatrix}1&1\\1&0\end{vmatrix}$	$\begin{vmatrix}0&1\\0&0\end{vmatrix}$
2	$\begin{vmatrix}0&1\\0&1\end{vmatrix}$ or $\begin{vmatrix}1&1\\1&1\end{vmatrix}$	$\begin{vmatrix}1&0\\1&0\end{vmatrix}$	8	$\begin{vmatrix}1&0\\0&1\end{vmatrix}$	$\begin{vmatrix}1&0\\0&1\end{vmatrix}$
3	$\begin{vmatrix}1&1\\0&0\end{vmatrix}$	$\begin{vmatrix}1&1\\0&0\end{vmatrix}$	9	$\begin{vmatrix}0&1\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\1&0\end{vmatrix}$
4	$\begin{vmatrix}0&0\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\1&1\end{vmatrix}$	10	$\begin{vmatrix}1&0\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\0&1\end{vmatrix}$
5	$\begin{vmatrix}1&0\\1&0\end{vmatrix}$ or $\begin{vmatrix}1&1\\1&1\end{vmatrix}$	$\begin{vmatrix}0&1\\0&1\end{vmatrix}$	11	$\begin{vmatrix}0&1\\1&0\end{vmatrix}$	$\begin{vmatrix}0&1\\1&0\end{vmatrix}$
6	$\begin{vmatrix}1&1\\0&1\end{vmatrix}$	$\begin{vmatrix}1&0\\0&0\end{vmatrix}$			

According to this tabulation, we rearrange $\mathfrak{D}(i)$, $1 \leqslant i \leqslant 11$, as

$$\mathfrak{D}'(1) = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \quad \mathfrak{D}'(2) = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}, \quad \mathfrak{D}'(3) = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}, \ldots, \mathfrak{D}'(11) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

Now, it is not difficult to follow the discussion in Section 6 and get $\langle \Gamma, \eta_T^2 \Gamma \rangle = \operatorname{tr.} \tilde{X}_1^m$, where

$$\tilde{X}_1 = \begin{pmatrix} U & 0 & 0 \\ 0 & W_1 & V_1 \\ 0 & V_1 & W_1 \end{pmatrix}, \qquad U = \begin{pmatrix} 4 & 8 & 6 & 6 & 4 \\ 1 & 8 & 6 & 6 & 6 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 7 & 12 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of $\tilde{X}_1$ are $1, -2, 2, 2, 2, a\ b, c, d, c, d$, where a, b are two roots of $x^2 - 12x + 24 = 0$ and c, d are two roots of $x^2 - 8x - 8 = 0$. As a result, we get

PROPOSITION 10.4.

$$\langle \Gamma\eta_T, \Gamma\eta_T \rangle = 1 + (-2)^m + 2^m + a^m + b^m + 2[2^m + c^m + d^m],$$

where a, b are two roots of $x^2 - 12x + 24 = 0$ and c, d are two roots of $x^2 - 8x - 8 = 0$.

Now we compute the term

$$\sum_{T' \neq I,\ J \in \Omega} (\Gamma\eta_T, \rho_I)(\Gamma\eta_T, \rho_J)\langle \Gamma\eta_{I'}, \Gamma\eta_{J'} \rangle \tag{10.5}$$

in (10.1). We note that Lemmas 7.8–7.12 are true for $G = SU_3(2^m)$. But for $G = SU_3(2^m)$, tabulation (7.13) should be changed to

	2 pattern of ρ_I	m type of ρ_I		2 pattern of ρ_I	m type of ρ_I
1	$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$	4	$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$
2	$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$	5	$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$
3	$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$	6	$\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$

By Lemmas 7.12(i), (ii), and 9.1, cases 4 and 5 occur for $G = SU_3(2^m)$. Also case 6 cannot occur for $G = SU_3(2^m)$ by Lemma 9.1 and the discussion following (7.17). Hence this tabulation can be reduced to

	1 pattern of ρ_I	m type of ρ_I
1	$\rho(w_1 + w_2)$	1
2	$\rho(w_2)$	$\rho(w_2)$
3	$\rho(w_1)$	$\rho(w_1)$

According to Lemma 9.1, Proposition 4.16 (under conditions (9.3) for $G = SU_3(2^m)$), and this tabulation, we rearrange $\mathfrak{R}(i)$, $1 \leqslant i \leqslant 12$, as

$$\begin{aligned}
&\mathfrak{R}'(1) = (\rho(w_1), \rho(w_1), \rho(w_1)), && \mathfrak{R}'(7) = (\rho(w_2), \rho(w_1), \rho(w_1 + w_2)),\\
&\mathfrak{R}'(2) = (\rho(w_1), \rho(w_1), \rho(w_2)), && \mathfrak{R}'(8) = (\rho(w_2), \rho(w_1), \rho(w_1)),\\
&\mathfrak{R}'(3) = (\rho(w_1), \rho(w_2), \rho(w_1 + w_2)), && \mathfrak{R}'(9) = (\rho(w_1), \rho(w_2), \rho(w_1)),\\
&\mathfrak{R}'(4) = (\rho(w_1), \rho(w_2), \rho(w_2)), && \mathfrak{R}'(10) = (\rho(w_1), \rho(w_1), \rho(w_1 + w_2)),\\
&\mathfrak{R}'(5) = (\rho(w_2), \rho(w_2), \rho(w_2)), && \mathfrak{R}'(11) = (\rho(w_2), \rho(w_1), \rho(w_2)),\\
&\mathfrak{R}'(6) = (\rho(w_2), \rho(w_2), \rho(w_1)), && \mathfrak{R}'(12) = (\rho(w_2), \rho(w_2), \rho(w_1 + w_2)).
\end{aligned}$$

Now, it is not difficult to follow the discussion in Section 7 and get that $(10.5) = \operatorname{tr} \tilde{X}_2^m$, where

$$\tilde{X}_2 = \begin{pmatrix} V_2 & U_2 & 0 & 0 \\ U_2 & V_2 & 0 & 0 \\ 0 & 0 & Y_2 & W_2 \\ 0 & 0 & W_2 & Y_2 \end{pmatrix},$$

and

$$U_2 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 6 & 0 & 0 & 6 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 4 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 2 & 0 \\ 6 & 3 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & 0 \\ 6 & 4 \end{pmatrix}.$$

The eigenvalues of $\tilde{X}_2$ are 2, 7, 1, -2, -2, -2, -2, -2, e, e, f, f, where e, f are two roots of $x^2 - 8x + 10 = 0$. As a result, we get

PROPOSITION 10.6.

$$\sum_{T' \neq I, J \in \Omega} (\Gamma\eta_T, \rho_I)(\Gamma\eta_T, \rho_J)\langle \Gamma\eta_{I'}, \Gamma\eta_{J'}\rangle$$
$$= 2^m + 7^m + 1 + 5(-2)^m + 2(e^m + f^m),$$

where e, f *are two roots of* $x^2 - 8x + 10 = 0$.

Now we consider the two terms

$$\sum_{T' \neq J \in \Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle + \sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle \tag{10.7}$$

in (10.1). As proved in Section 8, the two summations in (10.7) are equal. Hence we need only consider

$$\sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma, \eta_{I'}\eta_T\Gamma\rangle. \tag{10.8}$$

We note that Lemmas 8.9–8.13 are true for $G = SU_3(2^m)$. But for $G = SU_3(2^m)$, tabulation (8.14) should be changed to

	2 pattern of ρ_I	m type of ρ_I		2 pattern of ρ_I	m type of ρ_I
1	$\begin{vmatrix}1&1\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\0&0\end{vmatrix}$	5	$\begin{vmatrix}1&1\\1&0\end{vmatrix}$	$\begin{vmatrix}0&1\\0&0\end{vmatrix}$
2	$\begin{vmatrix}1&1\\0&1\end{vmatrix}$	$\begin{vmatrix}1&0\\0&0\end{vmatrix}$	6	$\begin{vmatrix}1&0\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\0&1\end{vmatrix}$
3	$\begin{vmatrix}0&1\\1&1\end{vmatrix}$	$\begin{vmatrix}0&0\\1&0\end{vmatrix}$	7	$\begin{vmatrix}0&1\\1&0\end{vmatrix}$	$\begin{vmatrix}0&1\\1&0\end{vmatrix}$
4	$\begin{vmatrix}0&1\\0&1\end{vmatrix}$ or $\begin{vmatrix}1&1\\1&1\end{vmatrix}$	$\begin{vmatrix}1&0\\1&0\end{vmatrix}$	8	$\begin{vmatrix}1&0\\0&1\end{vmatrix}$	$\begin{vmatrix}1&0\\0&1\end{vmatrix}$

By Lemmas 8.13 and 9.1, cases 7 and 8 in this tabulation cannot occur for $G = SU_3(2^m)$. According to the tabulation, we rearrange $\mathfrak{Y}(i)$, $1 \leqslant i \leqslant 12$, as

$$\mathfrak{Y}'(1) = \left(\rho(w_1), \begin{vmatrix}1&1\\1&1\end{vmatrix}\right), \qquad \mathfrak{Y}'(7) = \left(\rho(w_2), \begin{vmatrix}0&1\\0&1\end{vmatrix}\right),$$

$$\mathfrak{Y}'(2) = \left(\rho(w_1), \begin{vmatrix}0&1\\1&1\end{vmatrix}\right), \qquad \mathfrak{Y}'(8) = \left(\rho(w_2), \begin{vmatrix}1&0\\1&1\end{vmatrix}\right),$$

$$\mathfrak{Y}'(3) = \left(\rho(w_1), \begin{vmatrix}0&1\\0&1\end{vmatrix}\right), \qquad \mathfrak{Y}'(9) = \left(\rho(w_1), \begin{vmatrix}1&1\\0&1\end{vmatrix}\right),$$

$$\mathfrak{Y}'(4) = \left(\rho(w_1), \begin{vmatrix}1&1\\1&0\end{vmatrix}\right), \qquad \mathfrak{Y}'(10) = \left(\rho(w_1), \begin{vmatrix}1&0\\1&1\end{vmatrix}\right),$$

$$\mathfrak{Y}'(5) = \left(\rho(w_2), \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right), \qquad \mathfrak{Y}'(11) = \left(\rho(w_2), \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\right),$$

$$\mathfrak{Y}'(6) = \left(\rho(w_2), \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right), \qquad \mathfrak{Y}'(12) = \left(\rho(w_2), \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right).$$

Now we may follow the discussion in Section 8 and get that $(10.8) = \operatorname{tr} \tilde{X}_4^m$, where

$$\tilde{X}_4 = \begin{pmatrix} V_4 & U_4 & 0 & 0 \\ U_4 & V_4 & 0 & 0 \\ 0 & 0 & Y_4 & W_4 \\ 0 & 0 & W_4 & Y_4 \end{pmatrix},$$

and

$$U_4 = \begin{pmatrix} 0 & 4 & 0 & 2 \\ 2 & 0 & 10 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \qquad V_4 = \begin{pmatrix} 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W_4 = \begin{pmatrix} 4 & 4 \\ 0 & 2 \end{pmatrix}, \qquad Y_4 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\tilde{X}_4$ are -2, -2, α, β, γ, α, β, γ, 8, 2, 0, -2, where α, β, and γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$. As a result, we get

PROPOSITION 10.9.

$$\sum_{T' \neq J \in \Omega} (\Gamma\eta_T, \rho_J)\langle \Gamma\eta_T, \Gamma\eta_{J'}\rangle + \sum_{T' \neq I \in \Omega} (\Gamma\eta_T, \rho_I)\langle \Gamma\eta_{I'}, \Gamma\eta_T\rangle$$
$$= 2^{3m+1} + 2^{m+1} + 6(-2)^m + 4(\alpha^m + \beta^m + \gamma^m),$$

where α, β, γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$.

By (10.1), (10.3), and Propositions 10.4, 10.6, and 10.9, we get

THEOREM 10.10.

$$C_{\Phi,\Phi} = \langle \Phi_{(\phi,\phi)}, \Phi_{(\phi,\phi)}\rangle = 7^m + 1 - 2^{3m+1} + a^m + b^m$$
$$+ 2(2^m + c^m + d^m + e^m + f^m) - 4[\alpha^m + \beta^m + \gamma^m],$$

where a, b are two roots of $x^2 - 12x + 24 = 0$. c, d are two roots of $x^2 - 8x - 8 = 0$, e, f are two roots of $x^2 - 8x + 10 = 0$, and α, β, γ are three roots of $x^3 - 8x^2 + 2x + 12 = 0$.

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